

Charalambos D. Aliprantis
Kim C. Border

Infinite Dimensional Analysis

A Hitchhiker's Guide

Third Edition

**With 38 Figures
and 1 Table**

 **Springer**

Contents

Preface to the third edition	vii
A foreword to the practical	xix
1 Odds and ends	1
1.1 Numbers	1
1.2 Sets	2
1.3 Relations, correspondences, and functions	4
1.4 A bestiary of relations	5
1.5 Equivalence relations	7
1.6 Orders and such	7
1.7 Real functions	8
1.8 Duality of evaluation	9
1.9 Infinities	10
1.10 The Diagonal Theorem and Russell's Paradox	12
1.11 The axiom of choice and axiomatic set theory	13
1.12 Zorn's Lemma	15
1.13 Ordinals	18
2 Topology	21
2.1 Topological spaces	23
2.2 Neighborhoods and closures	26
2.3 Dense subsets	28
2.4 Nets	29
2.5 Filters	32
2.6 Nets and Fillers	35
2.7 Continuous functions	36
2.8 Compactness	38
2.9 Nets vs. sequences	41
2.10 Semicontinuous functions	43
2.11 Separation properties	44
2.12 Comparing topologies	47
2.13 Weak topologies	47
2.14 The product topology	50
2.15 Pointwise and uniform convergence	53

2.16	Locally compact spaces	55
2.17	The Stone-Cech compactification	58
2.18	Stone-Cech compactification of a discrete set	63
2.19	Paracompact spaces and partitions of unity	65
3	Metrizable spaces	69
3.1	Metric spaces	70
3.2	Completeness	73
3.3	Uniformly continuous functions	76
3.4	Semicontinuous functions on metric spaces	79
3.5	Distance functions	80
3.6	Embeddings and completions	84
3.7	Compactness and completeness	85
3.8	Countable products of metric spaces	89
3.9	The Hilbert cube and metrization	90
3.10	Locally compact metrizable spaces	92
3.11	The Baire Category Theorem	93
3.12	Contraction mappings	95
3.13	The Cantor set	98
3.14	The Baire space $\mathbb{N}^{\mathbb{N}}$	101
3.15	Uniformities	108
3.16	The Hausdorff distance	109
3.17	The Hausdorff metric topology	113
3.18	Topologies for spaces of subsets	119
3.19	The space $C(X, Y)$	123
4	Measurability	127
4.1	Algebras of sets	129
4.2	Rings and semirings of sets	131
4.3	Dynkin's lemma	135
4.4	The Borel σ -algebra	137
4.5	Measurable functions	139
4.6	The space of measurable functions	141
4.7	Simple functions	144
4.8	The σ -algebra induced by a function	147
4.9	Product structures	148
4.10	Caratheodory functions	153
4.11	Borel functions and continuity	156
4.12	The Baire σ -algebra	158
5	Topological vector spaces	163
5.1	Linear topologies	166
5.2	Absorbing and circled sets	168
5.3	Metrizable topological vector spaces	172
5.4	The Open Mapping and Closed Graph Theorems	175
5.5	Finite dimensional topological vector spaces	177

5.6	Convex sets	181
5.7	Convex and concave functions	186
5.8	Sublinear functions and gauges	190
5.9	The Hahn-Banach Extension Theorem	195
5.10	Separating hyperplane theorems	197
5.11	Separation by continuous functionals	201
5.12	Locally convex spaces and seminorms	204
5.13	Separation in locally convex spaces	207
5.14	Dual pairs	211
5.15	Topologies consistent with a given dual	213
5.16	Polars	215
5.17	\mathfrak{L} -topologies	220
5.18	The Mackey topology	223
5.19	The strong topology	223
6	Normed spaces	225
6.1	Normed and Banach spaces	227
6.2	Linear operators on normed spaces	229
6.3	The norm dual of a normed space	230
6.4	The uniform boundedness principle	232
6.5	Weak topologies on normed spaces	235
6.6	Metrizability of weak topologies	237
6.7	Continuity of the evaluation	241
6.8	Adjoint operators	243
6.9	Projections and the fixed space of an operator	244
6.10	Hilbert spaces	246
7	Convexity	251
7.1	Extended-valued convex functions	254
7.2	Lower semicontinuous convex functions	255
7.3	Support points	258
7.4	Subgradients	264
7.5	Supporting hyperplanes and cones	268
7.6	Convex functions on finite dimensional spaces	271
7.7	Separation and support in finite dimensional spaces	275
7.8	Supporting convex subsets of Hilbert spaces	280
7.9	The Bishop-Phelps Theorem	281
7.10	Support functionals	288
7.11	Support functionals and the Hausdorff metric	292
7.12	Extreme points of convex sets	294
7.13	Quasiconvexity	299
7.14	Polytopes and weak neighborhoods	300
7.15	Exposed points of convex sets	305

8	Riesz spaces	311
8.1	Orders, lattices, and cones	312
8.2	Riesz spaces	313
8.3	Order bounded sets	315
8.4	Order and lattice properties	316
8.5	The Riesz decomposition property	319
8.6	Disjointness	320
8.7	Riesz subspaces and ideals	321
8.8	Order convergence and order continuity	322
8.9	Bands	324
8.10	Positive functionals	325
8.11	Extending positive functionals	330
8.12	Positive operators	332
8.13	Topological Riesz spaces	334
8.14	The band generated by E'	339
8.15	Riesz pairs	340
8.16	Symmetric Riesz pairs	342
9	Banach lattices	347
9.1	Frechet and Banach lattices	348
9.2	The Stone-Weierstrass Theorem	352
9.3	Lattice homomorphisms and isometries	353
9.4	Order continuous norms	355
9.5	AM- and AL-spaces	357
9.6	The interior of the positive cone	362
9.7	Positive projections	364
9.8	The curious AL-space BV_0	365
10	Charges and measures	371
10.1	Set functions	374
10.2	Limits of sequences of measures	379
10.3	Outer measures and measurable sets	379
10.4	The Caratheodory extension of a measure	381
10.5	Measure spaces	387
10.6	Lebesgue measure	389
10.7	Product measures	391
10.8	Measures on \mathbb{R}^n	392
10.9	Atoms	395
10.10	The AL-space of charges	396
10.11	The AL-space of measures	399
10.12	Absolute continuity	401

11 Integrals	403
11.1 The integral of a step function	404
11.2 Finitely additive integration of bounded functions	406
11.3 The Lebesgue integral	408
11.4 Continuity properties of the Lebesgue integral	413
11.5 The extended Lebesgue integral	416
11.6 Iterated integrals	418
11.7 The Riemann integral	419
11.8 The Bochner integral	422
11.9 The Gelfand integral	428
11.10 The Dunford and Pettis integrals	431
12 Measures and topology	433
12.1 Borel measures and regularity	434
12.2 Regular Borel measures	438
12.3 The support of a measure	441
12.4 Nonatomic Borel measures	443
12.5 Analytic sets	446
12.6 The Choquet Capacity Theorem	456
13 L_p-spaces	461
13.1 L_p -norms	462
13.2 Inequalities of Holder and Minkowski	463
13.3 Dense subspaces of L_p -spaces	466
13.4 Sublattices of L_p -spaces	467
13.5 Separable L_1 -spaces and measures	468
13.6 The Radon-Nikodym Theorem	469
13.7 Equivalent measures	471
13.8 Duals of L_p -spaces	473
13.9 Lyapunov's Convexity Theorem	475
13.10 Convergence in measure	479
13.11 Convergence in measure in L_p -spaces	481
13.12 Change of variables	483
14 Riesz Representation Theorems	487
14.1 The AM-space $B_b(I)$ and its dual	488
14.2 The dual of $C_b(X)$ for normal spaces	491
14.3 The dual of $C_c(X)$ for locally compact spaces	496
14.4 Baire vs. Borel measures	498
14.5 Homomorphisms between $C(X)$ -spaces	500

15 Probability measures	505
15.1 The weak* topology on $\mathcal{Z}(X)$	506
15.2 Embedding X in $T(X)$	512
15.3 Properties of $\mathcal{Z}(X)$	513
15.4 The many faces of $\mathcal{Z}(X)$	517
15.5 Compactness in $\mathbb{C}\mathcal{P}(X)$	518
15.6 The Kolmogorov Extension Theorem	519
16 Spaces of sequences	525
16.1 The basic sequence spaces	526
16.2 The sequence spaces R^A and \mathfrak{P}	527
16.3 The sequence space c_0	529
16.4 The sequence space c	531
16.5 The C_p -spaces	533
16.6 Δ and the symmetric Riesz pair $(C_X, C\Delta)$	537
16.7 The sequence space	538
16.8 More on $C_X = ba(N)$	543
16.9 Embedding sequence spaces	546
16.10 Banach-Mazur limits and invariant measures	550
16.11 Sequences of vector spaces	552
17 Correspondences	555
17.1 Basic definitions	556
17.2 Continuity of correspondences	558
17.3 Hemicontinuity and nets	563
17.4 Operations on correspondences	566
17.5 The Maximum Theorem	569
17.6 Vector-valued correspondences	571
17.7 Demicontinuous correspondences	574
17.8 Knaster-Kuratowski-Mazurkiewicz mappings	577
17.9 Fixed point theorems	581
17.10 Contraction correspondences	585
17.11 Continuous selectors	587
18 Measurable correspondences	591
18.1 Measurability notions	592
18.2 Compact-valued correspondences as functions	597
18.3 Measurable selectors	600
18.4 Correspondences with measurable graph	606
18.5 Correspondences with compact convex values	609
18.6 Integration of correspondences	614

19 Markov transitions	621
19.1 Markov and stochastic operators	623
19.2 Markov transitions and kernels	625
19.3 Continuous Markov transitions	631
19.4 Invariant measures	631
19.5 Ergodic measures	636
19.6 Markov transition correspondences	638
19.7 Random functions	641
19.8 Dilations	645
19.9 More on Markov operators	650
19.10 A note on dynamical systems	652
20 Ergodicity	655
20.1 Measure-preserving transformations and ergodicity	656
20.2 Birkhoff's Ergodic Theorem	659
20.3 Ergodic operators	661
References	667
Index	681