

José Ferreirós

# **Labyrinth of Thought**

A History of Set Theory  
and Its Role in Modern Mathematics

Birkhäuser Verlag  
Basel · Boston · Berlin

# Contents

<b>Introduction</b> .....	xi
1. Aims and Scope .....	xiii
2. General Historiographical Remarks .....	xvii
 <b>Part One: The Emergence of Sets within Mathematics</b> .....	1
 <b>I Institutional and Intellectual Contexts in German Mathematics, 1800–1870</b> .....	3
1. Mathematics at the Reformed German Universities .....	4
2. Traditional and ‘Modern’ Foundational Viewpoints .....	10
3. The Issue of the Infinite .....	18
4. The Göttingen Group, 1855–1859 .....	24
5. The Berlin School, 1855–1870 .....	32
 <b>II A New Fundamental Notion: Riemann’s Manifolds</b> .....	39
1. The Historical Context: Grössenlehre, Gauss, and Herbart .....	41
2. Logical Prerequisites .....	47
3. The Mathematical Context of Riemann’s Innovation .....	53
4. Riemann’s General Definition .....	62
5. Manifolds, Arithmetic, and Topology .....	67
6. Riemann’s Influence on the Development of Set Theory .....	70
<i>Appendix: Riemann and Dedekind</i> .....	77
 <b>III Dedekind and the Set-theoretical Approach to Algebra</b> .....	81
1. The Algebraic Origins of Dedekind’s Set Theory, 1856–58 .....	82
2. A New Fundamental Notion for Algebra: Fields .....	90
3. The Emergence of Algebraic Number Theory .....	94

4. Ideals and Methodology .....	99
5. Dedekind's Infinitism .....	107
6. The Diffusion of Dedekind's Views .....	111
<b>IV The Real Number System .....</b>	<b>117</b>
1. 'Construction' vs. Axiomatization .....	119
2. The Definitions of the Real Numbers .....	124
3. The Influence of Riemann: Continuity in Arithmetic and Geometry .....	135
4. Elements of the Topology of $\mathbb{R}$ .....	137
<b>V Origins of the Theory of Point-Sets .....</b>	<b>145</b>
1. Dirichlet and Riemann: Transformations in the Theory of Real Functions .....	147
2. Lipschitz and Hankel on Nowhere Dense Sets and Integration .....	154
3. Cantor on Sets of the First Species .....	157
4. Nowhere Dense Sets of the Second Species .....	161
5. Crystallization of the Notion of Content .....	165
<b>Part Two: Entering the Labyrinth – Toward Abstract Set Theory .....</b>	<b>169</b>
<b>VI The Notion of Cardinality and the Continuum Hypothesis .....</b>	<b>171</b>
1. The Relations and Correspondence Between Cantor and Dedekind .....	172
2. Non-denumerability of $\mathbb{R}$ .....	176
3. Cantor's Exposition and the 'Berlin Circumstances' .....	183
4. Equipollence of Continua $\mathbb{R}$ and $\mathbb{R}'$ .....	187
5. Cantor's Difficulties .....	197
6. Derived Sets and Cardinalities .....	202
7. Cantor's Definition of the Continuum .....	208
8. Further Efforts on the Continuum Hypothesis .....	210
<b>VII Sets and Maps as a Foundation for Mathematics .....</b>	<b>215</b>
1. Origins of Dedekind's Program for the Foundations of Arithmetic .....	218
2. Theory of Sets, Mappings, and Chains .....	224

3. Through the Natural Numbers to Pure Mathematics .....	232
4. Dedekind and the Cantor–Bernstein Theorem .....	239
5. Dedekind’s Theorem of Infinity, and Epistemology .....	241
6. Reception of Dedekind’s Ideas .....	248
<b>VIII The Transfinite Ordinals and Cantor’s Mature Theory .....</b>	<b>257</b>
1. “Free Mathematics” .....	259
2. Cantor’s Notion of Set in the Early 1880s .....	263
3. The Transfinite (Ordinal) Numbers .....	267
4. Ordered Sets .....	274
5. The Reception in the Early 1880s .....	282
6. Cantor’s Theorem .....	286
7. The <i>Beiträge zur Begründung der transfiniten Mengenlehre</i> .....	288
8. Cantor and the Paradoxes .....	290
<b>Part Three: In Search of an Axiom System .....</b>	<b>297</b>
<b>IX Diffusion, Crisis, and Bifurcation: 1890 to 1914 .....</b>	<b>299</b>
1. Spreading Set Theory .....	300
2. The Complex Emergence of the Paradoxes .....	306
3. The Axiom of Choice and the Early Foundational Debate .....	311
4. The Early Work of Zermelo .....	317
5. Russell’s Theory of Types .....	325
6. Other Developments in Set Theory .....	333
<b>X Logic and Type Theory in the Interwar Period .....</b>	<b>337</b>
1. An Atmosphere of Insecurity: Weyl, Brouwer, Hilbert .....	338
2. Diverging Conceptions of Logic .....	345
3. The Road to the Simple Theory of Types .....	348
4. Type Theory at its Zenith .....	353
5. A Radical Proposal: Weyl and Skolem on First-Order Logic .....	357
<b>XI Consolidation of Axiomatic Set Theory .....</b>	<b>365</b>
1. The Contributions of Fraenkel .....	366

2. Toward the Modern Axiom System: von Neumann and Zermelo .....	370
3. The System von Neumann–Bernays–Gödel .....	378
4. Gödel’s Relative Consistency Results .....	382
5. First-Order Axiomatic Set Theory .....	386
6. A Glance Ahead: Mathematicians and Foundations after World War II .....	388
<b>Bibliographical References .....</b>	<b>393</b>
<b>Index of Illustrations .....</b>	<b>422</b>
<b>Name Index .....</b>	<b>423</b>
<b>Subject Index .....</b>	<b>430</b>