MARCH OF THE PARAMETER OF A REPAREMENT OF A REPAREMENT

Advanced Series on

Statistical Science &

Applied Probability

ESSENTIALS OF STOCHASTIC FINANCE Facts, Models, Theory

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