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Methods for Partial Differential Equations

Qualitative Properties of Solutions,
Phase Space Analysis, Semilinear Models

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