

# **Copula Methods in Finance**

**Umberto Cherubini  
Elisa Luciano**

and

**Walter Vecchiato**

**John Wiley & Sons, Ltd**

# Contents

Preface	xi
List of Common Symbols and Notations	xv
<b>1 Derivatives Pricing, Hedging and Risk Management: The State of the Art</b>	<b>1</b>
1.1 Introduction	1
1.2 Derivative pricing basics: the binomial model	2
1.2.1 Replicating portfolios	3
1.2.2 No-arbitrage and the risk-neutral probability measure	3
1.2.3 No-arbitrage and the objective probability measure	4
1.2.4 Discounting under different probability measures	5
1.2.5 Multiple states of the world	6
1.3 The Black-Scholes model	7
1.3.1 Ito's lemma	8
1.3.2 Girsanov theorem	9
1.3.3 The martingale property	11
1.3.4 Digital options	*
1.4 Interest rate derivatives	12
1.4.1 Affine factor models	13
1.4.2 Forward martingale measure	15
1.4.3 LIBOR market model	16
1.5 Smile and term structure effects of volatility	18
1.5.1 Stochastic volatility models	18
1.5.2 Local volatility models	19
1.5.3 Implied probability	20
1.6 Incomplete markets	21
1.6.1 Back to utility theory	22
1.6.2 Super-hedging strategies	23
1.7 Credit risk	27
1.7.1 Structural models	28
1.7.2 Reduced form models	31
1.7.3 Implied default probabilities	33

1.7.4	Counterparty risk	36
1.8	Copula methods in finance: a primer	37
1.8.1	Joint probabilities, marginal probabilities and copula functions	38
1.8.2	Copula functions duality	39
1.8.3	Examples of copula functions	39
1.8.4	Copula functions and market comovements	41
1.8.5	Tail dependence	42
1.8.6	Equity-linked products	43
1.8.7	Credit-linked products	44
<b>2</b>	<b>Bivariate Copula Functions</b>	<b>49</b>
2.1	Definition and properties	49
2.2	Frechet bounds and concordance order	52
2.3	Sklar's theorem and the probabilistic interpretation of copulas	56
2.3.1	Sklar's theorem	56
2.3.2	The subcopula in Sklar's theorem	59
2.3.3	Modeling consequences	60
2.3.4	Sklar's theorem in financial applications: toward a non-Black-Scholes world	61
2.4	Copulas as dependence functions: basic facts	70
2.4.1	Independence	70
2.4.2	Comonotonicity	70
2.4.3	Monotone transforms and copula invariance	72
2.4.4	An application: VaR trade-off	73
2.5	Survival copula and joint survival function	75
2.5.1	An application: default probability with exogenous shocks	78
2.6	Density and canonical representation	81
2.7	Bounds for the distribution functions of sum of r.v.s	84
2.7.1	An application: VaR bounds	85
2.8	Appendix	87
<b>3</b>	<b>Market Comovements and Copula Families</b>	<b>95</b>
3.1	Measures of association	95
3.1.1	Concordance	95
3.1.2	Kendall's $\tau$	97
3.1.3	Spearman's $\rho$	100
3.1.4	Linear correlation	103
3.1.5	Tail dependence	108
3.1.6	Positive quadrant dependency	110
3.^	Parametric families of bivariate copulas	112
3.2.1	The bivariate Gaussian copula	112
3.2.2	The bivariate Student's $t$ copula	116
3.2.3	The Frechet family	118
3.2.4	Archimedean copulas	120
3.2.5	The Marshall-Olkin copula	128

<b>4 Multivariate Copulas</b>	<b>129</b>
4.1 Definition and basic properties	129
4.2 Frechet bounds and concordance order: the multidimensional case	133
4.3 Sklar's theorem and the basic probabilistic interpretation: the multidimensional case	135
4.3.1 Modeling consequences	138
4.4 Survival copula and joint survival function	140
4.5 Density and canonical representation of a multidimensional copula	144
4.6 Bounds for distribution functions of sums of $n$ random variables	145
4.7 Multivariate dependence	146
4.8 Parametric families of $n$ -dimensional copulas	147
4.8.1 The multivariate Gaussian copula	147
4.8.2 The multivariate Student's $t$ copula	148
4.8.3 The multivariate dispersion copula	149
4.8.4 Archimedean copulas	149
<b>5 Estimation and Calibration from Market Data</b>	<b>153</b>
5.1 Statistical inference for copulas	153
5.2 Exact maximum likelihood method	154
5.2.1 Examples	155
5.3 IFM method	156
5.3.1 Application: estimation of the parametric copula for market data	158
5.4 CML method	160
5.4.1 Application: estimation of the correlation matrix for a Gaussian copula	160
5.5 Non-parametric estimation	161
5.5.1 The empirical copula	161
5.5.2 Kernel copula	162
5.6 Calibration method by using sample dependence measures	172
5.7 Application	174
5.8 Evaluation criteria for copulas	176
5.9 Conditional copula	177
5.9.1 Application to an equity portfolio	178
<b>6 Simulation of Market Scenarios</b>	<b>181</b>
6.1 Monte Carlo application with copulas	181
6.2 Simulation methods for elliptical copulas	181
6.3 Conditional sampling	182
6.3.1 Clayton $n$ -copula	184
6.3.2 Gumbel $n$ -copula	185
6.3.3 Frank $n$ -copula	186
6.4 Marshall and Olkin's method	188
6.5 Examples of simulations	191
<b>7 Credit Risk Applications</b>	<b>195</b>
7.1 Credit derivatives	195

7.2	Overview of some credit derivatives products	196
7.2.1	Credit default swap	196
7.2.2	Basket default swap	198
7.2.3	Other credit derivatives products	199
7.2.4	Collateralized debt obligation (CDO)	199
7.3	Copula approach	202
7.3.1	Review of single survival time modeling and calibration	202
7.3.2	Multiple survival times: modeling	203
7.3.3	Multiple defaults: calibration	205
7.3.4	Loss distribution and the pricing of CDOs	206
7.3.5	Loss distribution and the pricing of homogeneous basket default swaps	208
7.4	Application: pricing and risk monitoring a CDO	210
7.4.1	Dow Jones EuroStoxx50 CDO	210
7.4.2	Application: basket default swap	210
7.4.3	Empirical application for the EuroStoxx50 CDO	212
7.4.4	EuroStoxx50 pricing and risk monitoring	216
7.4.5	Pricing and risk monitoring of the basket default swaps	221
7.5	Technical appendix	225
7.5.1	Derivation of a multivariate Clayton copula density	225
7.5.2	Derivation of a 4-variate Frank copula density	226
7.5.3	Correlated default times	227
7.5.4	Variance-covariance robust estimation	228
7.5.5	Interest rates and foreign exchange rates in the analysis	229

## 8 Option Pricing with Copulas

8.1	Introduction	231
8.2	Pricing bivariate options in complete markets	232
8.2.1	Copula pricing kernels	232
8.2.2	Alternative pricing techniques	235
8.3	Pricing bivariate options in incomplete markets	239
8.3.1	Frechet pricing: super-replication in two dimensions	240
8.3.2	Copula pricing kernel	241
8.4	Pricing vulnerable options	243
8.4.1	Vulnerable digital options	244
8.4.2	Pricing vulnerable call options	246
8.4.3	Pricing vulnerable put options	248
8.4.4	Pricing vulnerable options in practice	250
8.5	Pricing rainbow two-color options	253
8.5.1	Call option on the minimum of two assets	254
8.5.2	Call option on the maximum of two assets	257
8.5.3	Put option on the maximum of two assets	258
8.5.4	Put option on the minimum of two assets	261
8.5.5	Option to exchange	262
8.5.6	Pricing and hedging rainbows with smiles: Everest notes	263
8.6	Pricing barrier options	267
8.6.1	Pricing call barrier options with copulas: the general framework	268

8.6.2	Pricing put barrier option: the general framework	270
8.6.3	Specifying the trigger event	272
8.6.4	Calibrating the dependence structure	276
8.6.5	The reflection copula	276
8.7	Pricing multivariate options: Monte Carlo methods	278
8.7.1	Application: basket option	279
	<b>Bibliography</b>	<b>281</b>
	<b>Index</b>	<b>289</b>