## Contents

*Preface*  
page xiii

**Introduction: motivating examples**  
0.1 Lifts of diffusion processes  
0.2 Markovian lifting of stochastic delay equations  
0.3 Zakai's equation  
0.4 Random motion of a string  
0.5 Stochastic equation of the free field  
0.6 Equation of stochastic quantization  
0.7 Reaction-diffusion equation  
0.8 An example arising in neurophysiology  
0.9 Equation of population genetics  
0.10 Musiela's equation of the bond market  

**PART ONE FOUNDATIONS**  

1 Random variables  
1.1 Random variables and their integrals  
1.2 Operator valued random variables  
1.3 Conditional expectation and independence  

2 Probability measures  
2.1 General properties  
2.2 Gaussian measures in Banach spaces  
2.2.1 Fernique theorem  
2.2.2 Reproducing kernels  
2.2.3 White noise expansions  
2.3 Probability measures on Hilbert spaces  
2.3.1 Gaussian measures on Hilbert spaces  
2.3.2 Feldman–Hajek theorem
2.3.3 An application to a general Cameron–Martin formula 59
2.3.4 The Bochner theorem 60

3 Stochastic processes 65
3.1 General concepts 65
3.2 Kolmogorov test 67
3.3 Processes with filtration 71
3.4 Martingales 73
3.5 Stopping times and Markov processes 77
3.6 Gaussian processes in Hilbert spaces 77
3.7 Stochastic processes as random variables 78

4 The stochastic integral 80
4.1 Wiener processes 80
  4.1.1 Hilbert space valued Wiener processes 81
  4.1.2 Generalized Wiener processes on a Hilbert space 84
  4.1.3 Wiener processes in \( U = L^2(\mathcal{H}) \) 86
  4.1.4 Spatially homogeneous Wiener processes 90
  4.1.5 Complements on a Brownian sheet 94
4.2 Definition of the stochastic integral 95
  4.2.1 Stochastic integral for generalized Wiener processes 100
  4.2.2 Approximations of stochastic integrals 102
4.3 Properties of the stochastic integral 103
4.4 The Itô formula 106
4.5 Stochastic Fubini theorem 110
4.6 Basic estimates 114
4.7 Remarks on generalization of the integral 117

PART TWO EXISTENCE AND UNIQUENESS 119

5 Linear equations with additive noise 121
5.1 Basic concepts 121
  5.1.1 Concept of solutions 121
  5.1.2 Stochastic convolution 123
5.2 Existence and uniqueness of weak solutions 125
5.3 Continuity of weak solutions 129
  5.3.1 Factorization formula 129
5.4 Regularity of weak solutions in the analytic case 134
  5.4.1 Basic regularity theorems 134
  5.4.2 Regularity in the border case 139
5.5 Regularity of weak solutions in the space of continuous functions 143
Contents

5.5.1 The case when \( A \) is self-adjoint 143
5.5.2 The case of a skew-symmetric generator 149
5.5.3 Equations with spatially homogeneous noise 150
5.6 Existence of strong solutions 156

6 Linear equations with multiplicative noise 159
6.1 Strong, weak and mild solutions 159
6.1.1 The case when \( B \) is bounded 164
6.2 Stochastic convolution for contraction semigroups 166
6.3 Stochastic convolution for analytic semigroups 168
6.3.1 General results 168
6.3.2 Variational case 171
6.3.3 Self-adjoint case 172
6.4 Maximal regularity for stochastic convolutions in \( L^p \) spaces 173
6.4.1 Maximal regularity 173
6.5 Existence of mild solutions in the analytic case 176
6.5.1 Introduction 176
6.5.2 Existence of solutions in the analytic case 176
6.6 Existence of strong solutions 181

7 Existence and uniqueness for nonlinear equations 186
7.1 Equations with Lipschitz nonlinearities 186
7.1.1 The case of cylindrical Wiener processes 196
7.2 Nonlinear equations on Banach spaces: additive noise 200
7.2.1 Locally Lipschitz nonlinearities 200
7.2.2 Dissipative nonlinearities 204
7.2.3 Dissipative nonlinearities by Euler approximations 207
7.2.4 Dissipative nonlinearities and general initial conditions 210
7.2.5 Dissipative nonlinearities and general noise 213
7.3 Nonlinear equations on Banach spaces: multiplicative noise 215
7.4 Strong solutions 218

8 Martingale solutions 220
8.1 Introduction 220
8.2 Representation theorem 222
8.3 Compactness results 226
8.4 Proof of the main theorem 229

PART THREE PROPERTIES OF SOLUTIONS 233

9 Markov property and Kolmogorov equation 235
9.1 Regular dependence of solutions on initial data 235
9.1.1 Differentiability with respect to the initial condition 238
9.1.2 Comments on stochastic flows 245
9.2 Markov and strong Markov properties 247
9.2.1 Case of Lipschitz nonlinearities 247
9.2.2 Markov property for equations in Banach spaces 252
9.3 Kolmogorov's equation: smooth initial functions 253
9.3.1 Bounded generators 254
9.3.2 Arbitrary generators 256
9.4 Further regularity properties of the transition semigroup 259
9.4.1 Linear case 259
9.4.2 Nonlinear case 266
9.5 Mild Kolmogorov equation 271
9.5.1 Solution of (9.75) 272
9.5.2 Identification of \( v(t, \cdot) \) with \( P_t \varphi \) 274
9.6 Specific examples 278

10 Absolute continuity and the Girsanov theorem 282
10.1 Absolute continuity for linear systems 282
10.1.1 The case \( B = \tilde{B} = I \) 287
10.2 Girsanov's theorem and absolute continuity for nonlinear systems 291
10.2.1 Girsanov's theorem 291
10.3 Application to weak solutions 296

11 Large time behavior of solutions 300
11.1 Basic concepts 300
11.2 The Krylov–Bogoliubov existence theorem 304
11.2.1 Mixing and recurrence 307
11.2.2 Regular, strong Feller and irreducible semigroups 307
11.3 Linear equations with additive noise 308
11.3.1 Characterization theorem 310
11.3.2 Uniqueness of the invariant measure and asymptotic behavior 313
11.3.3 Strong Feller case 314
11.4 Linear equations with multiplicative noise 317
11.4.1 Bounded diffusion operators 317
11.4.2 Unbounded diffusion operators 322
11.5 General linear equations 324
11.6 Dissipative systems 326
11.6.1 Regular coefficients 327
11.6.2 Discontinuous coefficients 328
11.7 The compact case 332
11.7.1 Finite trace Wiener processes 333
11.7.2 Cylindrical Wiener processes 336
12 Small noise asymptotic behavior 339
  12.1 Large deviation principle 339
    12.1.1 Formulation and basic properties 341
    12.1.2 Lower estimates 341
    12.1.3 Upper estimates 342
    12.1.4 Change of variables 343
  12.2 LDP for a family of Gaussian measures 344
  12.3 LDP for Ornstein–Uhlenbeck processes 347
  12.4 LDP for semilinear equations 350
  12.5 Exit problem 351
    12.5.1 Exit rate estimates 353
    12.5.2 Exit place determination 358
    12.5.3 Explicit formulae for gradient systems 363

13 Survey of specific equations 368
  13.1 Countable systems of stochastic differential equations 368
  13.2 Delay equations 369
  13.3 First order equations 369
  13.4 Reaction-diffusion equations 370
    13.4.1 Spatially homogeneous noise 370
    13.4.2 Skorohod equations in infinite dimensions 371
  13.5 Equations for manifold valued processes 372
  13.6 Equations with random boundary conditions 372
  13.7 Equation of stochastic quantization 373
  13.8 Filtering equations 375
  13.9 Burgers equations 375
  13.10 Kardar, Parisi and Zhang equation 376
  13.11 Navier–Stokes equations and hydrodynamics 377
    13.11.1 Existence and uniqueness for $d = 2$ 377
    13.11.2 Existence and uniqueness for $d = 3$ 378
    13.11.3 Stochastic magneto-hydrodynamics equations 379
    13.11.4 The tamed Navier–Stokes equation 380
    13.11.5 Renormalization of the Navier–Stokes equation 380
    13.11.6 Euler equations 380
  13.12 Stochastic climate models 380
  13.13 Quasi-geostrophic equation 381
  13.14 A growth of surface equation 381
  13.15 Geometric SPDEs 382
  13.16 Kuramoto–Sivashinsky equation 382
  13.17 Cahn–Hilliard equations 383
  13.18 Porous media equations 384
  13.19 Korteweg–de Vries equation 386
13.19.1 Existence and uniqueness 386
13.19.2 Soliton dynamic 386
13.20 Stochastic conservation laws 386
13.21 Wave equations 387
13.21.1 Spatially homogeneous noise 388
13.21.2 Symmetric hyperbolic systems 389
13.21.3 Wave equations in Riemannian manifolds 389
13.22 Beam equations 389
13.23 Nonlinear Schrödinger equations 390
13.23.1 Existence and uniqueness 390
13.23.2 Blow-up 391

14 Some recent developments 392
14.1 Complements on solutions of equations 392
14.1.1 Stochastic PDEs in Banach spaces 392
14.1.2 Backward stochastic differential equations 393
14.1.3 Wiener chaos expansions 395
14.1.4 Hida's white noise approach 395
14.1.5 Rough paths approach 396
14.1.6 Equations with fractional Brownian motion 398
14.1.7 Equations with Lévy noise 398
14.1.8 Equations with irregular coefficients 399
14.1.9 Yamada–Watanabe theory in infinite dimensions 399
14.1.10 Numerical methods for SPDEs 399
14.2 Some results on laws of solutions 400
14.2.1 Applications of Malliavin calculus 400
14.2.2 Fokker–Planck and mass transport equations 401
14.2.3 Ultraboundedness and Harnack inequalities 402
14.2.4 Gradient flows in Wasserstein spaces and Dirichlet forms 402
14.3 Asymptotic properties of the solutions 403
14.3.1 More on invariant measures 403
14.3.2 More on large deviations 404
14.3.3 Stochastic resonance 404
14.3.4 Averaging 404
14.3.5 Short time asymptotic 405

Appendix A Linear deterministic equations 406
A.1 Cauchy problems and semigroups 406
A.2 Basic properties of $C_0$-semigroups 407
A.3 Cauchy problem for nonhomogeneous equations 409
A.4 Cauchy problem for analytic semigroups 412
A.5 Example of deterministic systems 419


| Contents |
|----------|------|
| Appendix B Some results on control theory | 428 |
| B.1 Controllability and stabilizability | 428 |
| B.2 Comparison of images of linear operators | 429 |
| B.3 Operators associated with control systems | 431 |
| Appendix C Nuclear and Hilbert–Schmidt operators | 436 |
| Appendix D Dissipative mappings | 440 |
| D.1 Subdifferential of the norm | 440 |
| D.2 Dissipative mappings | 442 |
| D.3 Continuous dissipative mappings | 444 |

Bibliography 446
Index 491