A TABLE OF SERIES AND PRODUCTS

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Series (6.1.1) through (12.1.3) have summands which are rational, factorial, and power functions of the index of summation k and parameter(s). The following index for these series has a letter R and/or F and/or E in the first column according as k occurs rationally and/or factorially and/or exponentially. Similarly, the second column contains these letters to indicate the occurrence of parameter(s). If R occurs in parentheses, it is to be understood that the corresponding quantity may or may not occur rationally.

	Occurrence of k:	*Occurrence of Parameters:*	
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17.7	R, F	R, F	245
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