

4

NICOLAS BOURBAKI

Algebra II

Chapters 4-7

Translated by P.M. Cohn & J. Howie



Springer

Table of contents

CHAPTER IV. — POLYNOMIALS AND RATIONAL FRACTIONS	IV.1
§ 1. <i>Polynomials</i>	IV.1
1. Definition of polynomials	IV.1
2. Degrees	IV.2
3. Substitutions	IV.4
4. Differentials and derivations	IV.6
5. Divisors of zero in a polynomial ring	IV.9
6. Euclidean division of polynomials in one indeterminate	IV.10
7. Divisibility of polynomials in one indeterminate	IV.11
8. Irreducible polynomials	IV.13
§ 2. <i>Zeros of polynomials</i>	IV.14
1. Roots of a polynomial in one indeterminate. Multipli- city	IV.14
2. Differential criterion for the multiplicity of a root	IV.17
3. Polynomial functions on an infinite integral domain	IV.17
§ 3. <i>Rational fractions</i>	IV.19
1. Definition of rational fractions	IV.19
2. Degrees	IV.20
3. Substitutions	IV.21
4. Differentials and derivations	IV.23
§ 4. <i>Formal power series</i>	IV.24
1. Definition of formal power series. Order	IV.24
2. Topology on the set of formal power series. Summable families	IV.25
3. Substitutions	IV.28
4. Invertible formal power series	IV.30
5. Taylor's formula for formal power series	IV.31
6. Derivations in the algebra of formal power series	IV.32
7. The solution of equations in a formal power series ring	IV.35
8. Formal power series over an integral domain	IV.38
9. The field of fractions of the ring of formal power series in one indeterminate over a field	IV.38
10. Exponential and logarithm	IV.39

§ 5. <i>Symmetric tensors and polynomial mappings</i>	IV.41
1. Relative traces	IV.41
2. Definition of symmetric tensors	IV.42
3. Product for symmetric tensors	IV.43
4. Divided powers	IV.45
5. Symmetric tensors over a free module	IV.47
6. The functor TS	IV.48
7. Coproduct for symmetric tensors	IV.50
8. Relations between TS (M) and S (M)	IV.52
9. Homogeneous polynomial mappings	IV.54
10. Polynomial mappings	IV.57
11. Relations between S (M*), TS (M)* ^{gr} and Pol (M, A)	IV.59
§ 6. <i>Symmetric functions</i>	IV.61
1. Symmetric polynomials	IV.61
2. Symmetric rational fractions	IV.67
3. Symmetric formal power series	IV.67
4. Sums of powers	IV.70
5. Symmetric functions in the roots of a polynomial	IV.72
6. The resultant	IV.75
7. The discriminant	IV.81
Exercises on § 1	IV.86
Exercises on § 2	IV.87
Exercises on § 3	IV.89
Exercises on § 4	IV.90
Exercises on § 5	IV.91
Exercises on § 6	IV.98
Table	IV.103
CHAPTER V. — COMMUTATIVE FIELDS	V.1
§ 1. <i>Prime fields. Characteristic</i>	V.1
1. Prime fields	V.1
2. Characteristic of a ring and of a field	V.2
3. Commutative rings of characteristic p	V.3
4. Perfect rings of characteristic p	V.5
5. Characteristic exponent of a field. Perfect fields	V.7
6. Characterization of polynomials with zero differential ..	V.7
§ 2. <i>Extensions</i>	V.9
1. The structure of an extension	V.9
2. Degree of an extension	V.10
3. Adjunction	V.10
4. Composite extensions	V.12
5. Linearly disjoint extensions	V.13
§ 3. <i>Algebraic extensions</i>	V.15
1. Algebraic elements of an algebra	V.15

2. Algebraic extensions	V.17
3. Transitivity of algebraic extensions. Fields that are relatively algebraically closed in an extension field	V.19
§ 4. <i>Algebraically closed extensions</i>	V.19
1. Algebraically closed fields	V.19
2. Splitting extensions	V.21
3. Algebraic closure of a field	V.22
§ 5. <i>p-radical extensions</i>	V.24
1. <i>p</i> -radical elements	V.24
2. <i>p</i> -radical extensions	V.25
§ 6. <i>Etale algebra</i>	V.26
1. Linear independence of homomorphisms	V.26
2. Algebraic independence of homomorphisms	V.28
3. Diagonalizable algebras and etale algebras	V.28
4. Subalgebras of an etale algebra	V.30
5. Separable degree of a commutative algebra	V.31
6. Differential characterization of etale algebras	V.33
7. Reduced algebras and etale algebras	V.34
§ 7. <i>Separable algebraic extensions</i>	V.36
1. Separable algebraic extensions	V.36
2. Separable polynomials	V.37
3. Separable algebraic elements	V.39
4. The theorem of the primitive element	V.40
5. Stability properties of separable algebraic extensions ...	V.41
6. A separability criterion	V.42
7. The relative separable algebraic closure	V.43
8. The separable closure of a field	V.45
9. Separable and inseparable degrees of an extension of finite degree	V.46
§ 8. <i>Norms and traces</i>	V.47
1. Recall	V.47
2. Norms and traces in etale algebras	V.47
3. Norms and traces in extensions of finite degree	V.50
§ 9. <i>Conjugate elements and quasi-Galois extensions</i>	V.52
1. Extension of isomorphisms	V.52
2. Conjugate extensions. Conjugate elements	V.52
3. Quasi-Galois extensions	V.53
4. The quasi-Galois extension generated by a set	V.55
§ 10. <i>Galois extensions</i>	V.56
1. Definition of Galois extensions	V.56
2. The Galois group	V.58
3. Topology of the Galois group	V.60

4. Galois descent	V.62
5. Galois cohomology	V.64
6. Artin's theorem	V.65
7. The fundamental theorem of Galois theory	V.67
8. Change of base field	V.69
9. The normal basis theorem	V.72
10. Finite Γ -sets and étale algebras	V.75
11. The structure of quasi-Galois extensions	V.76
§ 11. <i>Abelian extensions</i>	V.77
1. Abelian extensions and the abelian closure	V.77
2. Roots of unity	V.78
3. Primitive roots of unity	V.79
4. Cyclotomic extensions	V.81
5. Irreducibility of cyclotomic polynomials	V.83
6. Cyclic extensions	V.85
7. Duality of $\mathbb{Z}/n\mathbb{Z}$ -modules	V.86
8. Kummer theory	V.88
9. Artin-Schreier theory	V.91
§ 12. <i>Finite fields</i>	V.93
1. The structure of finite fields	V.93
2. Algebraic extensions of a finite field	V.94
3. The Galois group of the algebraic closure of a finite field	V.96
4. Cyclotomic polynomials over a finite field	V.97
§ 13. <i>p-radical extensions of height ≤ 1</i>	V.98
1. p -free subsets and p -bases	V.98
2. Differentials and p -bases	V.100
3. The Galois correspondence between subfields and Lie algebras of derivations	V.104
§ 14. <i>Transcendental extensions</i>	V.106
1. Algebraically free families. Pure extensions	V.106
2. Transcendence bases	V.107
3. The transcendence degree of an extension	V.110
4. Extension of isomorphisms	V.111
5. Algebraically disjoint extensions	V.112
6. Algebraically free families of extensions	V.115
7. Finitely generated extensions	V.117
§ 15. <i>Separable extensions</i>	V.118
1. Characterization of the nilpotent elements of a ring	V.118
2. Separable algebras	V.119
3. Separable extensions	V.121
4. Mac Lane's separability criterion	V.122
5. Extensions of a perfect field	V.125
6. The characterization of separability by automorphisms	V.125

§ 16. <i>Differential criteria of separability</i>	V.127
1. Extension of derivations : the case of rings	V.127
2. Extension of derivations : the case of fields	V.128
3. Derivations in fields of characteristic zero	V.130
4. Derivations in separable extensions	V.131
5. The index of a linear mapping	V.132
6. Differential properties of finitely generated extensions .	V.133
7. Separating transcendence bases	V.136
§ 17. <i>Regular extensions</i>	V.137
1. Complements on the relative separable algebraic closure	V.137
2. The tensor product of extensions	V.139
3. Regular algebras	V.140
4. Regular extensions	V.141
5. Characterization of regular extensions	V.142
6. Application in composite extensions	V.143
Exercises on § 1	V.145
Exercises on § 2	V.146
Exercises on § 3	V.147
Exercises on § 4	V.150
Exercises on § 5	V.150
Exercises on § 6	V.151
Exercises on § 7	V.151
Exercises on § 8	V.153
Exercises on § 9	V.153
Exercises on § 10	V.154
Exercises on § 11	V.160
Exercises on § 12	V.166
Exercises on § 13	V.170
Exercises on § 14	V.171
Exercises on § 15	V.175
Exercises on § 16	V.177
Exercises on § 17	V.180
Historical note (chapters IV and V)	V.182
Bibliography	V.197
CHAPTER VI. — ORDERED GROUPS AND FIELDS	VI.1
§ 1. <i>Ordered groups. Divisibility</i>	VI.1
1. Definition of ordered monoids and groups	VI.1
2. Pre-ordered monoids and groups	VI.3
3. Positive elements	VI.3
4. Filtered groups	VI.4
5. Divisibility relations in a field	VI.5
6. Elementary operations on ordered groups	VI.7
7. Increasing homomorphisms of ordered groups	VI.7

8. Suprema and infima in an ordered group	VI.8
9. Lattice-ordered groups	VI.10
10. The decomposition theorem	VI.11
11. Positive and negative parts	VI.12
12. Coprime elements	VI.13
13. Irreducible elements	VI.17
§ 2. <i>Ordered fields</i>	VI.19
1. Ordered rings	VI.19
2. Ordered fields	VI.20
3. Extensions of ordered fields	VI.21
4. Algebraic extensions of ordered fields	VI.23
5. Maximal ordered fields	VI.25
6. Characterization of maximal ordered fields. Euler-Lagrange theorem	VI.26
7. Vector spaces over an ordered field	VI.28
Exercises on § 1	VI.30
Exercises on § 2	VI.37
CHAPTER VII. — MODULES OVER PRINCIPAL IDEAL DOMAINS	VII.1
§ 1. <i>Principal ideal domains</i>	VII.1
1. Definition of a principal ideal domain	VII.1
2. Divisibility in principal ideal domains	VII.1
3. Decomposition into irreducible factors in principal ideal domains	VII.3
4. Divisibility of rational integers	VII.5
5. Divisibility of polynomials in one indeterminate over a field	VII.5
§ 2. <i>Torsion modules over a principal ideal domain</i>	VII.6
1. Modules over a product of rings	VII.6
2. Canonical decomposition of a torsion module over a principal ideal domain	VII.7
3. Applications : I. Canonical decompositions of rational numbers and of rational functions in one indeterminate	VII.10
4. Applications : II. The multiplicative group of units of the integers modulo a	VII.12
§ 3. <i>Free modules over a principal ideal domain</i>	VII.14
§ 4. <i>Finitely generated modules over a principal ideal domain</i> ...	VII.15
1. Direct sums of cyclic modules	VII.15
2. Content of an element of a free module	VII.16
3. Invariant factors of a submodule	VII.18
4. Structure of finitely generated modules	VII.19
5. Calculation of invariant factors	VII.20

6. Linear mappings of free modules, and matrices over a principal ideal domain	VII.21
7. Finitely generated abelian groups	VII.22
8. Indecomposable modules. Elementary divisors	VII.23
9. Duality in modules of finite length over a principal ideal domain	VII.25

§ 5. <i>Endomorphisms of vector spaces</i>	VII.28
1. The module associated to an endomorphism	VII.28
2. Eigenvalues and eigenvectors	VII.30
3. Similarity invariants of an endomorphisms	VII.31
4. Triangularisable endomorphism	VII.34
5. Properties of the characteristic polynomial: trace and determinant	VII.36
6. Characteristic polynomial of the tensor product of two endomorphisms	VII.39
7. Diagonalisable endomorphisms	VII.40
8. Semi-simple and absolutely semi-simple endomorphisms	VII.41
9. Jordan decomposition	VII.43

Exercises on § 1	VII.48
Exercises on § 2	VII.54
Exercises on § 3	VII.59
Exercises on § 4	VII.62
Exercises on § 5	VII.70
Historical note (Chapters VI and VII)	VII.73
Bibliography	VII.83
Index of notations	445
Index of terminology	447
Table of contents	455