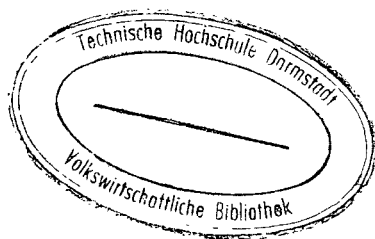


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# Markov-Switching Vector Autoregressions

Modelling, Statistical Inference,  
and Application to  
Business Cycle Analysis



Springer

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