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Representation Theory and Noncommutative Harmonic Analysis I

Fundamental Concepts. Representations of Virasoro
and Affine Algebras

With 11 Figures



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I. Introduction to the Theory of Representations and Noncommutative Harmonic Analysis

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