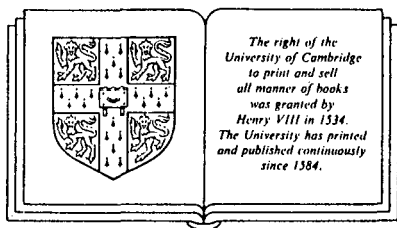


# Optimization and Stability Theory for Economic Analysis

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# CONTENTS

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Preface	<i>page</i> xiii
1 CONVEXITY	1
1.1 Introduction	1
1.2 Convex sets	1
1.2.1 Basic properties	1
1.2.2 Separation of convex sets	6
1.3 Convex and concave functions	16
1.3.1 Basic properties	16
1.3.2 Quasiconcave functions	25
1.3.3 Extremum properties	28
2 STATIC OPTIMIZATION	32
2.1 Introduction	32
2.2 Classical optimization	34
2.2.1 Unconstrained optimization	34
2.2.2 Equality constrained optimization	37
2.3 Non-linear programming	48
2.3.1 Saddlepoint optimization	48
2.3.2 Constraint qualifications	57
2.3.3 Quasiconcave programming	68
3 EQUILIBRIUM MATHEMATICS	73
3.1 Introduction	73
3.2 Equilibrium mathematics	73
3.2.1 Continuity of correspondences	74
3.2.2 Fixed-point theorems	84

## CONTENTS

3.3	Existence of competitive equilibrium	87
3.3.1	Introduction	87
3.3.2	Finite private ownership economy	88
3.3.3	Properties of demands and supplies	89
3.3.4	Equilibrium	92
3.4	Equilibrium in $n$ -person non-cooperative games	94
4	COMPARATIVE STATICS AND DUALITY	98
4.1	Introduction	98
4.2	Comparative statics and maximum value functions	99
4.3	An introduction to duality theory	117
5	DYNAMICS AND STABILITY	134
5.1	Introduction	134
5.2	Differential equations: basic concepts	135
5.2.1	Existence and uniqueness theorems	135
5.2.2	Equilibrium and stability concepts	141
5.3	Linear differential equations	144
5.4	Stability of linear differential equations	151
5.4.1	Necessary and sufficient conditions for stability	151
5.4.2	Necessary conditions for stability	154
5.4.3	Sufficient conditions for stability	154
5.5	Trajectories, conditional stability and saddlepoint equilibria	155
5.5.1	Trajectories	155
5.5.2	The stable manifold	156
5.5.3	Conditional stability	159
5.5.4	Identifying the stable manifold	160
5.6	Stability analysis for non-linear systems	162
5.6.1	The linear approximation method	163
5.6.2	The Olech sufficient conditions for global stability	166
5.6.3	Liapunov's second method	167
5.6.4	Non-unique equilibria	176
5.6.5	Phase diagrams	179

## CONTENTS

5.7	Economic applications	185
5.7.1	A resource market model	185
5.7.2	Stability of general equilibrium	189
5.7.3	A partly rational expectations macro model	198
5.8	Dynamical systems: topological considerations	203
5.8.1	Dynamical systems: introduction	203
5.8.2	The system $\dot{x} = f(x)$ , $x \in X \subset \mathbb{R}^2$	206
5.8.3	The system $\dot{x} = f(x)$ , $x \in X \subset \mathbb{R}^n$	223
5.9	Difference equation systems	226
5.9.1	Introduction	226
5.9.2	The linear difference equation system	228
5.9.3	Equilibrium and stability for linear systems	229
6	INTRODUCTION TO DYNAMIC OPTIMIZATION AND THE CALCULUS OF VARIATIONS	233
6.1	Introduction	233
6.2	Mathematical preliminaries	238
6.3	Separability and the principle of optimality	240
6.4	Calculus of variations: necessary conditions	241
6.4.1	First order necessary conditions	242
6.4.2	Second order necessary conditions	250
6.4.3	Weierstrass necessary condition	251
6.5	Variable endpoints and transversality conditions	252
6.6	The bequest or final value function	259
6.7	Corners	260
6.8	Summary of results for $x(t) \in \mathbb{R}^n$	262
6.8.1	Fixed endpoint problem	263
6.8.2	Variable endpoint problem	264
6.9	Sufficient conditions for finite horizon problems	265
6.10	Constraints in the calculus of variations	267
6.10.1	Global, integral constraints	267
6.10.2	Local constraints	269

## CONTENTS

6.11 An extended example	272
6.12 Infinite horizon problems	279
<b>7 OPTIMAL CONTROL THEORY</b>	<b>288</b>
7.1 Introduction	288
7.1.1 A variational approach to necessary conditions	288
7.1.2 Transversality conditions	293
7.1.3 Interpretation of the multipliers	294
7.2 Necessary conditions	296
7.3 Pontryagin's maximum principle	305
7.3.1 The control problem	305
7.3.2 Behaviour of the co-state variables	308
7.3.3 Spatial variations	310
7.4 Constraints on state and control variables	315
7.5 Sufficient conditions: finite horizon problems	327
7.6 Bounded state variable constraints	330
7.7 An extended example	335
7.8 Infinite horizon problems	346
<b>APPENDICES – MATHEMATICAL REVIEW</b>	<b>358</b>
Sets and mappings	358
Real and complex number fields	361
Vector spaces	364
Matrices	368
Linear equations	374
Eigenvalues	374
Quadratic forms	378
Further set properties	380
Continuity concepts	382
Differentiability	388
Taylor's theorem	393
Homogeneous functions	394
Implicit function theorem	394
Integration	395

## CONTENTS

<i>Glossary</i>	399
<i>Bibliography</i>	401
<i>Author Index</i>	408
<i>Subject Index</i>	410