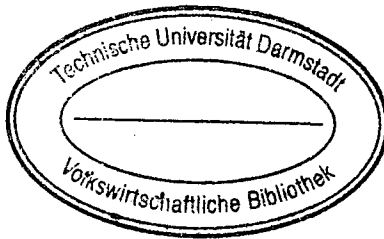


# Convergence of Probability Measures

## Second Edition

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\* Starred topics can be omitted on a first reading

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