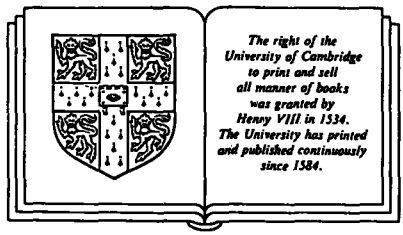
FOURIER ANALYSIS

T.W. KÖRNER

Trinity Hall, Cambridge



CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE

NEW YORK NEW ROCHELLE

MELBOURNE SYDNEY

CONTENTS

PREFACE *page* xi

PART I FOURIER SERIES 1

1 Introduction 3
2 Proof of Fejér's theorem 6
3 Weyl's equidistribution theorem 11
4 The Weierstrass polynomial approximation theorem 15
5 A second proof of Weierstrass's theorem 19
6 Hausdorff's moment problem 21
7 The importance of linearity 24
8 Compass and tides 28
9 The simplest convergence theorem 32
10 The rate of convergence 35
11 A nowhere differentiable function 38
12 Reactions 42
13 Monte Carlo methods 46
14 Mathematical Brownian motion 50
15 Pointwise convergence 56
16 Behaviour at points of discontinuity I 59
17 Behaviour at points of discontinuity II 62
18 A Fourier series divergent at a point 67
19 Pointwise convergence, the answer 74

PART II SOME DIFFERENTIAL EQUATIONS 77

20 The undisturbed damped oscillator does not explode 79
21 The disturbed damped linear oscillator does not explode 83
22 Transients 88
23 The linear damped oscillator with periodic input 93
24 A non-linear oscillator I 99
25 A non-linear oscillator II 104

26 A non-linear oscillator III 113
27 Poisson summation 116
28 Dirichlet's problem for the disc 121
29 Potential theory with smoothness assumptions 124
30 An example of Hadamard 131
31 Potential theory without smoothness assumptions 134

PART III ORTHOGONAL SERIES 143

32 Mean square approximation I 145
33 Mean square approximation II 150
34 Mean square convergence 155
35 The isoperimetric problem I 159
36 The isoperimetric problem II 166
37 The Sturm–Liouville equation I 170
38 Liouville 175
39 The Sturm–Liouville equation II 179
40 Orthogonal polynomials 185
41 Gaussian quadrature 191
42 Linkages 197
43 Tchebychev and uniform approximation I 201
44 The existence of the best approximation 207
45 Tchebychev and uniform approximation II 212

PART IV FOURIER TRANSFORMS 219

46 Introduction 221
47 Change in the order of integration I 226
48 Change in the order of integration II 230
49 Fejér's theorem for Fourier transforms 240
50 Sums of independent random variables 245
51 Convolution 253
52 Convolution on $\mathbb{T}$ 259
53 Differentiation under the integral 265
54 Lord Kelvin 270
55 The heat equation 274
56 The age of the earth I 282
57 The age of the earth II 285
58 The age of the earth III 289
59 Weierstrass's proof of Weierstrass's theorem 292
60 The inversion formula 295
61 Simple discontinuities 300
62 Heat flow in a semi-infinite rod 308

63 A second approach 315
64 The wave equation 324
65 The transatlantic cable I 332
66 The transatlantic cable II 335
67 Uniqueness for the heat equation I 338
68 Uniqueness for the heat equation II 344
69 The law of errors 347
70 The central limit theorem I 349
71 The central limit theorem II 357

PART V FURTHER DEVELOPMENTS 363

72 Stability and control 365
73 Instability 368
74 The Laplace transform 372
75 Deeper properties 379
76 Poles and stability 386
77 A simple time delay equation 395
78 An exception to a rule 403
79 Many dimensions 407
80 Sums of random vectors 413
81 A chi squared test 418
82 Haldane on fraud 425
83 An example of outstanding statistical treatment I 429
84 An example of outstanding statistical treatment II 434
85 An example of outstanding statistical treatment III 436
86 Will a random walk return? 443
87 Will a Brownian motion return? 451
88 Analytic maps of Brownian motion 455
89 Will a Brownian motion tangle? 461
90 La Famille Picard va á Monte Carlo 467

PART VI OTHER DIRECTIONS 471

91 The future of mathematics viewed from 1800 473
92 Who was Fourier? I 475
93 Who was Fourier? II 478
94 Why do we compute? 481
95 The diameter of stars 484
96 What do we compute? 488
97 Fourier analysis on the roots of unity 491
98 How do we compute? 497
99 How fast can we multiply? 500

100 What makes a good code? 503
101 A little group theory 506
102 A good code? 509
103 A little more group theory 513
104 Fourier analysis on finite Abelian groups 519
105 A formula of Euler 525
106 An idea of Dirichlet 532
107 Primes in some arithmetical progressions 539
108 Extension from real to complex variable 546
109 Primes in general arithmetical progressions 552
110 A word from our founder 558

Appendix A: The circle $\mathbb{T}$ 560

Appendix B: Continuous function on closed bounded sets 563

Appendix C: Weakening hypotheses 565

Appendix D: Ode to a galvanometer 575

Appendix E: The principle of the argument 577

Appendix F: Chase the constant 580

Appendix G: Are share prices in Brownian motion? 581

Index 585