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# Introductory Optimization Dynamics

Optimal Control with Economics and  
Management Applications

Second, Revised and Enlarged Edition

**Springer-Verlag**

Berlin Heidelberg New York

London Paris Tokyo

Hong Kong Barcelona

Budapest

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