

GARCH Models

Structure, Statistical Inference
and Financial Applications

Christian Francq

University Lille 3, Lille, France

Jean-Michel Zakoi'an

CREST, Paris, and University Lille 3, France

® WILEY

A John Wiley and Sons, Ltd., Publication

Contents

Preface	xi
Notation	xiii
1 Classical Time Series Models and Financial Series	1
1.1 Stationary Processes	1
1.2 ARMA and ARIMA Models	3
1.3 Financial Series	7
1.4 Random Variance Models	10
1.5 Bibliographical Notes	12
1.6 Exercises	12
Part I Univariate GARCH Models	17
2 GARCH(p, q) Processes	19
2.1 Definitions and Representations	19
2.2 Stationarity Study	24
2.2.1 The GARCH(1, 1) Case	24
2.2.2 The General Case	28
2.3 ARCH (∞) Representation*	39
2.3.1 Existence Conditions	39
2.3.2 ARCH (∞) Representation of a GARCH	42
2.3.3 Long-Memory ARCH	43
2.4 Properties of the Marginal Distribution	45
2.4.1 Even-Order Moments	45
2.4.2 Kurtosis	48
2.5 Autocovariances of the Squares of a GARCH	50
2.5.1 Positivity of the Autocovariances	50
2.5.2 The Autocovariances Do Not Always Decrease	51
2.5.3 Explicit Computation of the Autocovariances of the Squares	52
2.6 Theoretical Predictions	53
2.7 Bibliographical Notes	57
2.8 Exercises	58

3	Mixing*	63
3.1	Markov Chains with Continuous State Space	63
3.2	Mixing Properties of GARCH Processes	68
3.3	Bibliographical Notes	76
3.4	Exercises	76
4	Temporal Aggregation and Weak GARCH Models	79
4.1	Temporal Aggregation of GARCH Processes	79
4.1.1	Nontemporal Aggregation of Strong Models	80
4.1.2	Nonaggregation in the Class of Semi-Strong GARCH Processes	81
4.2	Weak GARCH	82
4.3	Aggregation of Strong GARCH Processes in the Weak GARCH Class	85
4.4	Bibliographical Notes	88
4.5	Exercises	89
Part II Statistical Inference		91
5	Identification	93
5.1	Autocorrelation Check for White Noise	93
5.1.1	Behavior of the Sample Autocorrelations of a GARCH Process	94
5.1.2	Portmanteau Tests	97
5.1.3	Sample Partial Autocorrelations of a GARCH	97
5.1.4	Numerical Illustrations	98
5.2	Identifying the ARMA Orders of an ARMA-GARCH	100
5.2.1*	Sample Autocorrelations of an ARMA-GARCH	101
5.2.2	Sample Autocorrelations of an ARMA-GARCH Process When the Noise is Not Symmetrically Distributed	104
5.2.3	Identifying the Orders (P, Q)	106
5.3	Identifying the GARCH Orders of an ARMA-GARCH Model	108
5.3.1	Corner Method in the GARCH Case	109
5.3.2	Applications	109
5.4	Lagrange Multiplier Test for Conditional Homoscedasticity	111
5.4.1	General Form of the LM Test	111
5.4.2	LM Test for Conditional Homoscedasticity	115
5.5	Application to Real Series	117
5.6	Bibliographical Notes	120
5.7	Exercises	122
6	Estimating ARCH Models by Least Squares	127
6.1	Estimation of ARCH(ω) models by Ordinary Least Squares	127
6.2	Estimation of ARCH(g) Models by Feasible Generalized Least Squares	132
6.3	Estimation by Constrained Ordinary Least Squares	135
6.3.1	Properties of the Constrained OLS Estimator	135
6.3.2	Computation of the Constrained OLS Estimator	137
6.4	Bibliographical Notes	138
6.5	Exercises	138
7	Estimating GARCH Models by Quasi-Maximum Likelihood	141
7.1	Conditional Quasi-Likelihood	141
7.1.1	Asymptotic Properties of the QMLE	143
7.1.2	The ARCH(1) Case: Numerical Evaluation of the Asymptotic Variance	147

7.1.3	The Nonstationary ARCH(1)	148
7.2	Estimation of ARMA-GARCH Models by Quasi-Maximum Likelihood	150
7.3	Application to Real Data	155
7.4	Proofs of the Asymptotic Results*	156
7.5	Bibliographical Notes	180
7.6	Exercises	180
8	Tests Based on the Likelihood	185
8.1	Test of the Second-Order Stationarity Assumption	186
8.2	Asymptotic Distribution of the QML When θ_0 is at the Boundary	187
8.2.1	Computation of the Asymptotic Distribution	191
8.3	Significance of the GARCH Coefficients	194
8.3.1	Tests and Rejection Regions	194
8.3.2	Modification of the Standard Tests	196
8.3.3	Test for the Nullity of One Coefficient	197
8.3.4	Conditional Homoscedasticity Tests with ARCH Models	199
8.3.5	Asymptotic Comparison of the Tests	201
8.4	Diagnostic Checking with Portmanteau Tests	204
8.5	Application: Is the GARCH(1,1) Model Overrepresented?	204
8.6	Proofs of the Main Results*	207
8.7	Bibliographical Notes	215
8.8	Exercises	215
9	Optimal Inference and Alternatives to the QMLE*	219
9.1	Maximum Likelihood Estimator	219
9.1.1	Asymptotic Behavior	220
9.1.2	One-Step Efficient Estimator	222
9.1.3	Semiparametric Models and Adaptive Estimators	223
9.1.4	Local Asymptotic Normality	226
9.2	Maximum Likelihood Estimator with Misspecified Density	231
9.2.1	Condition for the Convergence of $\hat{\theta}_N$, to θ_0	231
9.2.2	Reparameterization Implying the Convergence of $\hat{\theta}_N$, to θ_0	232
9.2.3	Choice of Instrumental Density h	233
9.2.4	Asymptotic Distribution of $B_{n,j}$,	234
9.3	Alternative Estimation Methods	236
9.3.1	Weighted LSE for the ARMA Parameters	236
9.3.2	Self-Weighted QMLE	237
9.3.3	L_p Estimators	237
9.3.4	Least Absolute Value Estimation	238
9.3.5	Whittle Estimator	238
9.4	Bibliographical Notes	239
9.5	Exercises	239

Part III Extensions and Applications 243

10	Asymmetries	245
10.1	Exponential GARCH Model	246
10.2	Threshold GARCH Model	250
10.3	Asymmetric Power GARCH Model	256
10.4	Other Asymmetric GARCH Models	258
10.5	A GARCH Model with Contemporaneous Conditional Asymmetry	259

10.6	Empirical Comparisons of Asymmetric GARCH Formulations	261
10.7	Bibliographical Notes	269
10.8	Exercises	270
11	Multivariate GARCH Processes	273
11.1	Multivariate Stationary Processes	273
11.2	Multivariate GARCH Models	275
11.2.1	Diagonal Model	276
11.2.2	Vector GARCH Model	277
11.2.3	Constant Conditional Correlations Models	279
11.2.4	Dynamic Conditional Correlations Models	281
11.2.5	BEKK-GARCH Model	281
11.2.6	Factor GARCH Models	284
11.3	Stationarity	286
11.3.1	Stationarity of VEC and BEKK Models	286
11.3.2	Stationarity of the CCC Model	289
11.4	Estimation of the CCC Model	291
11.4.1	Identifiability Conditions	292
11.4.2	Asymptotic Properties of the QMLE of the CCC-GARCH model	294
11.4.3	Proof of the Consistency and the Asymptotic Normality of the QML	296
11.5	Bibliographical Notes	307
11.6	Exercises	308
12	Financial Applications	311
12.1	Relation between GARCH and Continuous-Time Models	311
12.1.1	Some Properties of Stochastic Differential Equations	311
12.1.2	Convergence of Markov Chains to Diffusions	313
12.2	Option Pricing	319
12.2.1	Derivatives and Options	319
12.2.2	The Black-Scholes Approach	319
12.2.3	Historic Volatility and Implied Volatilities	321
12.2.4	Option Pricing when the Underlying Process is a GARCH	321
12.3	Value at Risk and Other Risk Measures	327
12.3.1	Value at Risk	327
12.3.2	Other Risk Measures	331
12.3.3	Estimation Methods	334
12.4	Bibliographical Notes	337
12.5	Exercises	338
Part IV Appendices		341
A	Ergodicity, Martingales, Mixing	343
A.1	Ergodicity	343
A.2	Martingale Increments	344
A.3	Mixing	347
A.3.1	α -Mixing and β -Mixing Coefficients	348
A.3.2	Covariance Inequality	349
A.3.3	Central Limit Theorem	352
B	Autocorrelation and Partial Autocorrelation	353
B.1	Partial Autocorrelation	353
B.2	Generalized Bartlett Formula for Nonlinear Processes	359

C Solutions to the Exercises	365
D Problems	439
References	473
Index	487