

INVERSE AND ILL-POSED PROBLEMS SERIES

Operator Theory and Ill-Posed Problems

M.M. Lavrent'ev and L.Ya. Savel'ev

///VSP///

LEIDEN • BOSTON

2006

Contents

	BASIC CONCEPTS	1
Chapter 1. Set theory		2
1.1. Sets		2
1.1.1. Elements and subsets		2
1.1.2. The algebra of sets		3
1.1.3. Cartesian product		5
1.2. Correspondences		7
1.2.1. Images and inverse images		7
1.2.2. Functions		8
1.2.3. Collections of sets		11
1.3. Relations		14
1.3.1. Reflexivity, transitivity, and symmetry		14
1.3.2. Equivalence		15
1.3.3. Order		17
1.4. Induction		23
1.4.1. Well-ordered sets		23
1.4.2. Discrete sets		24
1.4.3. Zorn's lemma		26
1.5. Natural numbers		27
1.5.1. Decimal natural numbers		27
1.5.2. The isomorphism theorem		28
1.5.3. Countable sets		29

Chapter 2. Algebra	30
2.1. Abstract algebra	30
2.1.1. Semigroups	30
2.1.2. Groups	37
2.1.3. Rings and fields	55
2.1.4. Lattices	64
2.1.5. Numbers	69
2.2. Linear algebra	75
2.2.1. Vector spaces	76
2.2.2. Linear operators	89
2.2.3. Linear functionals	102
2.2.4. Scalar products	114
2.2.5. Normed spaces	121
2.2.6. Euclidean spaces	130
2.3. Multilinear algebra	141
2.3.1. Tensor product	141
2.3.2. Exterior product	146
Chapter 3. Calculus	150
3.1. Limit	150
3.1.1. Topological spaces	150
3.1.2. Directed sets	170
3.1.3. Convergence	177
3.2. Differential	197
3.2.1. The definition of the differential	197
3.2.2. Differentiation rules	206
3.2.3. Lagrange's theorem	210
3.2.4. Termwise differentiation	215
3.2.5. Total differentials	217
3.2.6. Solution of functional equations	226
3.2.7. Taylor's formula	241
3.2.8. Local minima	250
3.2.9. Smooth curves	258
3.2.10. A simplest variational problem	261

3.3. Integral	263
3.3.1. Measures	264
3.3.2. Classical definition of the integral	272
3.3.3. Limit theorems	289
3.3.4. Measurable functions	295
3.3.5. The Fubini and Tonelli theorems	298
3.3.6. Indefinite integrals	316
3.4. Analysis on manifolds	321
3.4.1. Manifolds	321
3.4.2. The rank theorem	326
3.4.3. Sard's theorem	327
3.4.4. Differential forms	328
3.4.5. The Poincare theorem	331
3.4.6. Change of variables	334
3.4.7. Integral over a manifold	335
3.4.8. The Stokes formula	342
3.4.9. Map degree	348
3.4.10. Applications	351

OPERATORS 353

Chapter 4. Linear operators	354
4.1. Hilbert spaces	354
4.1.1. Orthogonal projection	354
4.1.2. Continuous linear functionals	356
4.1.3. The spaces $\mathcal{L}^2 = \mathcal{L}^2(U, \mu)$	359
4.2. Fourier series	363
4.2.1. Fourier coefficients	363
4.2.2. Isomorphism of Hilbert spaces	368
4.3. Function spaces	369
4.3.1. Metric spaces	369
4.3.2. Smooth functions	371
4.3.3. Lebesgue spaces	376
4.3.4. Distributions	380
4.3.5. Sobolev spaces	388

4.4. Fourier transform	391
4.4.1. Transforms of rapidly decreasing functions	391
4.4.2. Transforms of slowly increasing distributions	393
4.4.3. The Fourier–Plancherel transform	395
4.4.4. The Fourier–Stieltjes transform	395
4.4.5. The Radon transform	396
4.5. Bounded linear operators	397
4.5.1. Extensions of functionals	397
4.5.2. Uniform boundedness of operators	399
4.5.3. Inversion of operators	401
4.5.4. Closedness of the graph of an operator	403
4.5.5. Weak compactness	405
4.6. Compact linear operators	408
4.6.1. Examples of compact operators	408
4.6.2. Properties of compact operators	410
4.6.3. Adjoint operators	411
4.6.4. Fredholm operators	414
4.6.5. Fredholm theorems	417
4.7. Self-adjoint operators	419
4.7.1. Banach adjoint operators	419
4.7.2. Hilbert adjoint operators	420
4.7.3. Hermitian and normal operators	422
4.7.4. Unitary operators	423
4.7.5. Positive operators	424
4.8. Spectra of operators	426
4.8.1. Classification of spectra	426
4.8.2. The spectrum of a closed operator	431
4.8.3. The spectrum of a bounded operator	433
4.8.4. The spectrum of a compact operator	435
4.8.5. The spectrum of a self-adjoint operator	435
4.9. Spectral theorem	444
4.9.1. Projection measures	445
4.9.2. Integrals of bounded functions	451
4.9.3. Integrals of unbounded functions	459
4.9.4. Spectral theorem	462
4.9.5. Operator functions	466

4.10. Operator exponential	468
4.10.1. Problem formulation	468
4.10.2. Semigroups of operators	470
4.10.3. The Laplace transform	475
4.10.4. Stone's theorem	476
4.10.5. Evolution equations	478
Chapter 5. Nonlinear operators	483
5.1. Fixed points	483
5.1.1. The Brouwer theorem	483
5.1.2. The Tikhonov theorem and the Schauder theorem . . .	487
5.2. Saddle points	490
5.2.1. Kakutani's theorem	490
5.2.2. von Neumann theorem	493
5.3. Monotonic operators	497
5.3.1. Definition and properties	497
5.3.2. Equations with monotonic operators	499
5.4. Nonlinear contractions	501
5.4.1. Contracting semigroups of operators	501
5.4.2. Approximation	503
5.5. Degree theory	504
5.5.1. Finite-dimensional spaces	504
5.5.2. The Leray–Schauder degree	508
ILL-POSED PROBLEMS	511
Chapter 6. Classic problems	512
6.1. Mathematical description of the laws of physics	512
6.2. Equations of the first order	518
6.3. Classification of differential equations of the second order . . .	519
6.4. Elliptic equations	521
6.5. Hyperbolic and parabolic equations	527
6.6. The notion of well-posedness	529

Chapter 7. Ill-posed problems	531
7.1. Ill-posed Cauchy problems	531
7.2. Analytic continuation and interior problems	534
7.3. Weakly and strongly ill-posed problems. Problems of differentiation.	536
7.4. Reducing ill-posed problems to integral equations	537
Chapter 8. Physical problems leading to ill-posed problems	541
8.1. Interpretation of measurement data from physical devices . . .	541
8.2. Interpretation of gravimetric data	543
8.3. Problems for the diffusion equation	546
8.4. Determining physical fields from the measurements data . . .	547
8.5. Tomography	548
Chapter 9. Operator and integral equations	552
9.1. Definitions of well-posedness	552
9.2. Regularization	555
9.3. Linear operator equations	559
9.4. Integral equations with weak singularities	564
9.5. Scalar Volterra equations	565
9.6. Volterra operator equations	568
Chapter 10. Evolution equations	571
10.1. Cauchy problem and semigroups of operators	571
10.2. Equations in a Hilbert space	573
10.3. Equations with variable operator	577
10.4. Equations of the second order	578
10.5. Well-posed and ill-posed Cauchy problems	580
10.6. Equations with integro-differential operators	581
Chapter 11. Problems of integral geometry	584
11.1. Statement of problems of integral geometry	584
11.2. The Radon problem	584
11.3. Reconstructing a function from spherical means	588
11.4. Planar problem of the general form	594
11.5. Spatial problems of the general form	602

11.6. Problems of the Volterra type for manifolds invariant with respect to the translation group	614
11.7. Planar problems of integral geometry with a perturbation . .	618
Chapter 12. Inverse problems	626
12.1. Statement of inverse problems	626
12.2. Inverse dynamic problem. A linearization method	628
12.3. A general method for studying inverse problems for hyperbolic equations	637
12.4. The connection between inverse problems for hyperbolic, elliptic, and parabolic equations	644
12.5. Problems of determining a Riemannian metric	651
Chapter 13. Several areas of the theory of ill-posed problems, inverse problems, and applications	659
Bibliography	662
Index	673