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# Infinite Dimensional Analysis

A Hitchhiker's Guide

Second, Completely Revised  
and Enlarged Edition

With 21 Figures  
and 1 Table



Springer

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