

Richard Courant Fritz John

Introduction to Calculus and Analysis

Volume II

With the assistance of
Albert A. Blank and Alan Solomon

With 120 Illustrations



Springer

Contents

Chapter 1 Functions of Several Variables and Their Derivatives

1.1 Points and Points Sets in the Plane and in Space	1
a. Sequences of points. Convergence, 1 b. Sets of points in the plane, 3 c. The boundary of a set. Closed and open sets, 6 d. Closure as set of limit points, 9 e. Points and sets of points in space, 9	
1.2 Functions of Several Independent Variables	11
a. Functions and their domains, 11 b. The simplest types of functions, 12 c. Geometrical representation of functions, 13	
1.3 Continuity	17
a. Definition, 17 b. The concept of limit of a function of several variables, 19 c. The order to which a function vanishes, 22	
1.4 The Partial Derivatives of a Function	26
a. Definition. Geometrical representation, 26 b. Examples, 32 c. Continuity and the existence of partial derivatives, 34	

	d. Change of the order of differentiation, 36	
1.5	The Differential of a Function and Its Geometrical Meaning	40
	a. The concept of differentiability, 40 b. Directional derivatives, 43 c. Geometric interpretation of differentiability, The tangent plane, 46 d. The total differential of a function, 49 e. Application to the calculus of errors, 52	
1.6	Functions of Functions (Compound Functions) and the Introduction of New Independent Variables	53
	a. Compound functions. The chain rule, 53 b. Examples, 59 c. Change of independent variables, 60	
1.7	The Mean Value Theorem and Taylor's Theorem for Functions of Several Variables	64
	a. Preliminary remarks about approximation by polynomials, 64 b. The mean value theorem, 66 c. Taylor's theorem for several independent variables, 68	
1.8	Integrals of a Function Depending on a Parameter	71
	a. Examples and definitions, 71 b. Continuity and differentiability of an integral with respect to the parameter, 74 c. Interchange of integrations. Smoothing of functions, 80	
1.9	Differentials and Line Integrals	82
	a. Linear differential forms, 82	

- b. Line integrals of linear differential forms, 85 c. Dependence of line integrals on endpoints, 92

1.10 The Fundamental Theorem on Integrability of Linear Differential Forms	95
a. Integration of total differentials, 95	
b. Necessary conditions for line integrals to depend only on the end points, 96	
c. Insufficiency of the integrability conditions, 98	
d. Simply connected sets, 102	
e. The fundamental theorem, 104	

APPENDIX

A.1. The Principle of the Point of Accumulation in Several Dimensions and Its Applications	107
a. The principle of the point of accumulation, 107	
b. Cauchy's convergence test. Compactness, 108	
c. The Heine-Borel covering theorem, 109	
d. An application of the Heine-Borel theorem to closed sets contains in open sets, 110.	
A.2. Basic Properties of Continuous Functions	112
A.3. Basic Notions of the Theory of Point Sets	113
a. Sets and sub-sets, 113	
b. Union and intersection of sets, 115	
c. Applications to sets of points in the plane, 117.	
A.4. Homogeneous functions.	119

Chapter 2 Vectors, Matrices, Linear Transformations

2.1	Operations with Vectors	122
	a. Definition of vectors, 122	
	b. Geometric representation of vectors, 124	
	c. Length of vectors. Angles between directions, 127	
	d. Scalar products of vectors, 131	
	e. Equation of hyperplanes in vector form, 133	
	f. Linear dependence of vectors and systems of linear equations, 136	
2.2	Matrices and Linear Transformations	143
	a. Change of base. Linear spaces, 143	
	b. Matrices, 146	
	c. Operations with matrices, 150	
	d. Square matrices. The reciprocal of a matrix. Orthogonal matrices. 153	
2.3	Determinants	159
	a. Determinants of second and third order, 159	
	b. Linear and multilinear forms of vectors, 163	
	c. Alternating multilinear forms. Definition of determinants, 166	
	d. Principal properties of determinants, 171	
	e. Application of determinants to systems of linear equations. 175	
2.4	Geometrical Interpretation of Determinants	180
	a. Vector products and volumes of parallelepipeds in three-dimensional space, 180	
	b. Expansion of a determinant with respect to a column. Vector products in higher dimensions, 187	
	c. Areas of parallelograms and volumes of parallelepipeds in	

higher dimensions, 190 d. Orientation of parallelepipeds in n -dimensional space, 195 e. Orientation of planes and hyperplanes, 200 f. Change of volume of parallelepipeds in linear transformations, 201

2.5	Vector Notions in Analysis	204
	a. Vector fields, 204 b. Gradient of a scalar, 205 c. Divergence and curl of a vector field, 208 d. Families of vectors. Application to the theory of curves in space and to motion of particles, 211	

Chapter 3 Developments and Applications of the Differential Calculus

3.1	Implicit Functions	218
	a. General remarks, 218 b. Geometrical interpretation, 219 c. The implicit function theorem, 221 d. Proof of the implicit function theorem, 225 e. The implicit function theorem for more than two independent variables, 228	
3.2	Curves and Surfaces in Implicit Form	230
	a. Plane curves in implicit form, 230 b. Singular points of curves, 236 c. Implicit representation of surfaces, 238	
3.3	Systems of Functions, Transformations, and Mappings	241
	a. General remarks, 241 b. Curvilinear coordinates, 246 c. Extension to more than two independent variables, 249 d. Differentiation formulae for the inverse functions,	

	252	e. Symbolic product of mappings,	
	257	f. General theorem on the inversion of transformations and of systems of implicit functions. Decomposition into primitive mappings, 261	
		g. Alternate construction of the inverse mapping by the method of successive approximations, 266	
		h. Dependent functions, 268	
		i. Concluding remarks, 275	
3.4		Applications	278
		a. Elements of the theory of surfaces, 278	
		b. Conformal transformation in general, 289	
3.5		Families of Curves, Families of Surfaces, and Their Envelopes	290
		a. General remarks, 290	
		b. Envelopes of one-parameter families of curves, 292	
		c. Examples, 296	
		d. Endvelopes of families of surfaces, 303	
3.6		Alternating Differential Forms	307
		a. Definition of alternating differential forms, 307	
		b. Sums and products of differential forms, 310	
		c. Exterior derivatives of differential forms, 312	
		d. Exterior differential forms in arbitrary coordinates, 316	
3.7		Maxima and Minima	325
		a. Necessary conditions, 325	
		b. Examples, 327	
		c. Maxima and minima with subsidiary conditions, 330	
		d. Proof of the method of undetermined multipliers in the simplest case, 334	
		e. Generalization of the method of undetermined multipliers, 337	
		f. Examples, 340	

APPENDIX

A.1 Sufficient Conditions for Extreme Values	345
A.2 Numbers of Critical Points Re- lated to Indices of a Vector Field	352
A.3 Singular Points of Plane Curves	360
A.4 Singular Points of Surfaces	362
A.5 Connection Between Euler's and Lagrange's Representation of the motion of a Fluid	363
A.6 Tangential Representation of a Closed Curve and the Isoperi- metric Inequality	365

Chapter 4 Multiple Integrals

4.1 Areas in the Plane	367
a. Definition of the Jordan meas- ure of area, 367 b. A set that does not have an area, 370 c. Rules for operations with areas, 372	
4.2 Double Integrals	374
a. The double integral as a volume, 374 b. The general anal- ytic concept of the integral, 376 c. Examples, 379 d. Notation. Extensions. Fundamental rules, 381 e. Integral estimates and the mean value theorem, 383	
4.3 Integrals over Regions in three and more Dimensions	385

4.4	Space Differentiation. Mass and Density	386
4.5	Reduction of the Multiple Integral to Repeated Single Integrals	388
	a. Integrals over a rectangle, 388	
	b. Change of order of integration. Differentiation under the integral sign, 390	
	c. Reduction of double integrals to single integrals for more general regions, 392	
	d. Extension of the results to regions in several dimensions, 397	
4.6	Transformation of Multiple Integrals	398
	a. Transformation of integrals in the plane, 398	
	b. Regions of more than two dimensions, 403	
4.7	Improper Multiple Integrals	406
	a. Improper integrals of functions over bounded sets, 407	
	b. Proof of the general convergence theorem for improper integrals, 411	
	c. Integrals over unbounded regions, 414	
4.8	Geometrical Applications	417
	a. Elementary calculation of volumes, 417	
	b. General remarks on the calculation of volumes. Solids of revolution. Volumes in spherical coordinates, 419	
	c. Area of a curved surface, 421	
4.9	Physical Applications	431
	a. Moments and center of mass, 431	
	b. Moments of inertia, 433	
	c. The compound pendulum, 436	
	d. Potential of attracting masses, 438	

4.10 Multiple Integrals in Curvilinear Coordinates	445
<p>a. Resolution of multiple integrals, 445 b. Application to areas swept out by moving curves and volumes swept out by moving surfaces. Guldin's formula. The polar planimeter, 448</p>	
4.11 Volumes and Surface Areas in Any Number of Dimensions	453
<p>a. Surface areas and surface integrals in more than three dimensions, 453 b. Area and volume of the n-dimensional sphere, 455 c. Generalizations. Parametric Representations, 459</p>	
4.12 Improper Single Integrals as Functions of a Parameter	462
<p>a. Uniform convergence. Continuous dependence on the parameter, 462 b. Integration and differentiation of improper integrals with respect to a parameter, 466 c. Examples, 469 d. Evaluation of Fresnel's integrals, 473</p>	
4.13 The Fourier Integral	476
<p>a. Introduction, 476 b. Examples, 479 c. Proof of Fourier's integral theorem, 481 d. Rate of convergence in Fourier's integral theorem, 485 e. Parseval's identity for Fourier transforms, 488 f. The Fourier transformation for functions of several variables, 490</p>	
4.14 The Eulerian Integrals (Gamma Function)	497
<p>a. Definition and functional equa-</p>	

tion, 497 **b.** Convex functions. Proof of Bohr and Mollerup's theorem, 499 **c.** The infinite products for the gamma function, 503 **d.** The nextensio theorem, 507 **e.** The beta function, 508 **f.** Differentiation and integration of fractional order. Abel's integral equation, 511

APPENDIX: DETAILED ANALYSIS OF THE PROCESS OF INTEGRATION

A.1 Area	515
a. Subdivisions of the plane and the corresponding inner and outer areas, 515 b. Jordan-measurable sets and their areas, 517 c. Basic properties of areas, 519	
A.2 Integrals of Functions of Several Variables	524
a. Definition of the integral of a function $f(x, y)$, 524 b. Integrability of continuous functions and integrals over sets, 526 c. Basic rules for multiple integrals, 528 d. Reduction of multiple integrals to repeated single integrals, 531	
A.3 Transformation of Areas and Integrals	534
a. Mappings of sets, 534 b. Transformation of multiple integrals, 539	
A.4 Note on the Definition of the Area of a Curved Surface	540

Chapter 5 Relations Between Surface and Volume Integrals

5.1	Connection Between Line Integrals and Double Integrals in the Plane (The Integral Theorems of Gauss, Stokes, and Green)	543
5.2	Vector Form of the Divergence Theorem. Stokes's Theorem	551
5.3	Formula for Integration by Parts in Two Dimensions. Green's Theorem	556
5.4	The Divergence Theorem Applied to the Transformation of Double Integrals	558
	a. The case of 1-1 mappings, 558	
	b. Transformation of integrals and degree of mapping, 561	
5.5	Area Differentiation. Transformation of Δu to Polar Coordinates	565
5.6	Interpretation of the Formulae of Gauss and Stokes by Two-Dimensional Flows	569
5.7	Orientation of Surfaces	575
	a. Orientation of two-dimensional surfaces in three-space, 575	
	b. Orientation of curves on oriented surfaces, 587	
5.8	Integrals of Differential Forms and of Scalars over Surfaces	589
	a. Double integrals over oriented plane regions, 589	
	b. Surface	

integrals of second-order differential forms, 592 c. Relation between integrals of differential forms over oriented surfaces to integrals of scalars over unoriented surfaces, 594

5.9 Gauss's and Green's Theorems in Space 597

a. Gauss's theorem, 597 b. Application of Gauss's theorem to fluid flow, 602 c. Gauss's theorem applied to space forces and surface forces, 605 d. Integration by parts and Green's theorem in three dimensions, 607 e. Application of Green's theorem to the transformation of ΔU to spherical coordinates, 608

5.10 Stokes's Theorem in Space 611

a. Statement and proof of the theorem, 611 b. Interpretation of Stokes's theorem, 615

5.11 Integral Identities in Higher Dimensions 622

APPENDIX: GENERAL THEORY OF SURFACES AND OF SURFACE INTEGRALS

A.1 Surfaces and Surface Integrals in Three dimensions 624

a. Elementary surfaces, 624 b. Integral of a function over an elementary surface, 627 c. Oriented elementary surfaces, 629 d. Simple surfaces, 631 e. Partitions of unity and integrals over simple surfaces, 634

A.2 The Divergence Theorem	637
a. Statement of the theorem and its invariance, 637	
b. Proof of the theorem, 639	
A.3 Stokes's Theorem	642
A.4 Surfaces and Surface Integrals in Euclidean Spaces of Higher Dimensions	645
a. Elementary surfaces, 645	
b. Integral of a differential form over an oriented elementary surface, 647	
c. Simple m-dimensional surfaces, 648	
A.5 Integrals over Simple Surfaces, Gauss's Divergence Theorem, and the General Stokes Formula in Higher Dimensions	651

Chapter 6 Differential Equations

6.1 The Differential Equations for the Motion of a Particle in Three Dimensions	654
a. The equations of motion, 654	
b. The principle of conservation of energy, 656	
c. Equilibrium. Stability, 659	
d. Small oscillations about a position of equilibrium, 661	
e. Planetary motion, 665	
f. Boundary value problems. The loaded cable and the loaded beam, 672	
6.2 The General Linear Differential Equation of the First Order	678
a. Separation of variables, 678	
b. The linear first-order equation, 680	

6.3	Linear Differential Equations of Higher Order	683
	a. Principle of superposition. General solutions, 683 b. Homogeneous differential equations of the second order, 688 c. The non-homogeneous differential equations. Method of variation of parameters, 691	
6.4	General Differential Equations of the First Order	697
	a. Geometrical interpretation, 697 b. The differential equation of a family of curves. Singular solutions. Orthogonal trajectories, 699 c. Theorem of the existence and uniqueness of the solution, 702	
6.5	Systems of Differential Equations and Differential Equations of Higher Order	709
6.6	Integration by the Method of Undermined Coefficients	711
6.7	The Potential of Attracting Charges and Laplace's Equation	713
	a. Potentials of mass distributions, 713 b. The differential equation of the potential, 718 c. Uniform double layers, 719 d. The mean value theorem, 722 e. Boundary value problem for the circle. Poisson's integral, 724	
6.8	Further Examples of Partial Differential Equations from Mathematical Physics	727
	a. The wave equation in one dimension, 727 b. The wave equation	

in three-dimensional space, 728
 c. Maxwell's equations in free space,
 731

Chapter 7 Calculus of Variations

7.1 Functions and Their Extrema	737
7.2 Necessary conditions for Extreme Values of a Functional	741
a. Vanishing of the first variation, 741	
b. Deduction of Euler's differential equation, 743	
c. Proofs of the fundamental lemmas, 747	
d. Solution of Euler's differential equation in special cases. Examples, 748	
e. Identical vanishing of Euler's expression, 752	
7.3 Generalizations	753
a. Integrals with more than one argument function, 753	
b. Examples, 755	
c. Hamilton's principle. Lagrange's equations, 757	
d. Integrals involving higher derivatives, 759	
e. Several independent variables, 760	
7.4 Problems Involving Subsidiary Conditions. Lagrange Multipliers	762
a. Ordinary subsidiary conditions, 762	
b. Other types of subsidiary conditions, 765	

Chapter 8 Functions of a Complex Variable

8.1 Complex Functions Represented by Power Series	769
a. Limits and infinite series with complex terms, 769	
b. Power	

	series, 772	c. Differentiation and integration of power series, 773	
	d. Examples of power series, 776		
8.2	Foundations of the General Theory of Functions of a Complex Variable		778
	a. The postulate of differentiability, 778	b. The simplest operations of the differential calculus, 782	
	c. Conformal transformation. Inverse functions, 785		
8.3	The Integration of Analytic Functions		787
	a. Definition of the integral, 787		
	b. Cauchy's theorem, 789	c. Applications. The logarithm, the exponential function, and the general power function, 792	
8.4	Cauchy's Formula and Its Applications		797
	a. Cauchy's formula, 797	b. Expansion of analytic functions in power series, 799	
	c. The theory of functions and potential theory, 802		
	d. The converse of Cauchy's theorem, 803	e. Zeros, poles, and residues of an analytic function, 803	
8.5	Applications to Complex Integration (Contour Integration)		807
	a. Proof of the formula (8.22), 807		
	b. Proof of the formula (8.22), 808	c. Application of the theorem of residues to the integration of rational functions, 809	
	d. The theorem of residues and linear differential equations with constant coefficients, 812		

8.6 Many-Valued Functions and Analytic Extension	814
---	-----

<i>List of Biographical Dates</i>	941*
-----------------------------------	------

<i>Index</i>	943**
--------------	-------

* page 543 of this edition

** page 545 of this edition