

Karl Vind

Independence, Additivity, Uncertainty

With Contributions
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With 10 Figures



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¹Letter of May, 1983 to Paul Samuelson, copy to me summer 2000.

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