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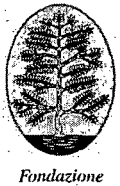
Optimal Transportation and Applications

Lectures given at the
C.I.M.E. Summer School
held in Martina Franca, Italy,
September 2–8, 2001

Editors: L. A. Caffarelli
S. Salsa



Fondazione
C.I.M.E.

Springer

Mathematik-Bibliothek

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