# Differential Topology and General Equilibrium with Complete and Incomplete Markets

#### by

**Antonio Villanacci** Universita degli Studi di Firenze, Italy

Laura Carosi Universita degli Studi di Pisa, Italy

### Pierluigi Benevieri

Universita degli Studi di Firenze, Italy

and

Andrea Battinelli Universita degli Studi di Siena, Italy



## Contents

List of Figures	XI
Acknowledgments	XV
Introduction	xvii

### Parti

1.	PREREQUISITES		3	
	1 Notation and symbols			4
	2	Miscel	laneous results	9
		2.1	Functions and sets	9
		2.2	Calculus	9
		2.3	Linear algebra	9
		2.4	General topology	11
	3	Nonlin	ear programming	12
			Concavity and generalized concavity	12
		.1	Concavity	12
		.2	Strict concavity	13
		.3	Quasiconcavity	13
		.4	Strict quasiconcavity	14
		.5	Relationships among kinds of concavity	16
		3.2	Constrained optimization	16
		3.2.1	The case of equality constraints	16
		3.2.2	The case of inequality constraints	18
		3.2.3	The case of equality and inequality constraints	19
		Our no	tation for rank computations	21
	4.1 Perturbing rows			

		4.2	Forgetting lower rows	23
		4.3	A step-by-step procedure	25
2.	MA	NIFOLD	DS IN EUCLIDEAN SPACES	29
	1	Manifo	olds - Definition and examples	32
	2	Appen	dix - Abstract manifolds	46
3.	DIF	FEREN	ΓIALS	49
	1	Introdu	uction	49
	2	Tanger	nt spaces	51
	3	Definit	tion of differential	57
	4	Differe	entials and local behavior	66
		4.1	Preliminaries	66
		4.2	The inverse function theorem for manifolds	66
		4.3	Immersions and embeddings	68
		4.4	Submersions	73
	5	Appen theorem	ndix - $C^r$ functions on abstract manifolds and Whitney's m	77
4.	REC	GULAR	VALUES	79
	1	Introd	uction	79
	2	Definit	tions	79
	3	Inverse	e images of regular values	84
		3.1	The regular value theorem	84
		3.2	Regular values and nonlinear systems of equations	87
		3.3 A	partial converse of the regular value theorem and the normal bundle	88
	4	Proper	functions	90
	5	Local	properness	96
	6	Two applications		101
		6.1	Submersiveness of functions related to the equilibrium manifold	101
		6.2	The Lagrange multiplier rule	105
5.	MANIFOLDS WITH BOUNDARY			
	1	1 Manifolds with boundary		107
	2	Interse	ection and inverse images of manifolds: the notion of	
		transve	ersality	120
		2.1	Intersecting manifolds	121
		2.2	Transversality	122

_				
6.	SARI	D'S TH	IEOREM AND TRANSVERSALITY	133
	1	Introdu	action	133
	2	Sard's	theorem	133
		2.1	Sets of (Lebesgue) measure zero	134
		2.2	Sard's theorem for functions between euclidean spaces	142
		2.3	Sard's theorem for functions between manifolds	149
	3	A trans	sversality result	151
	4	Full an	d zero measure in convex sets	153
7.	HOM	10TOP	Y AND DEGREE THEORY	159
	1	Introdu	action	159
	2	Homotopies		160
	3 Degree theory mod 2		e theory mod 2	163
		3.1	Definition of degree for $C^2$ functions at a regular value	165
		3.1.1	Definition and examples	165
		3.1.2	Properties of the degree	169
		3.2	Definition of degree for $C^2$ functions at an arbitrary	
			value	177
		3.3	Extension to continuous functions	184
	4	Comp	utation of degree for C <sup>1</sup> functions	197
	5	Existence results for nonlinear equations		
		5.1	Case A: manifolds in euclidean spaces	199
		5.2	Case B: abstract manifolds	200
	6	Appen	dix	201

vii

### Part II

8.	EXCHANGE ECONOMIES		207
	1 11	e model	208
	1.1	The commodity space	208
	1.2	Households	209
	1.2	.1 Consumption sets	209
	1.2	.2 Preferences	210
	1.2	.3 Utility functions	211
	1.2	.4 Initial endowments	214
	1.2	.5 Prices	214
	1.2	.6 Competitive budget set	215
	De	finition of equilibrium	217

	3	The der	mand function	220
	4	The ext	tended approach	223
	5	Pareto	optimal allocations	225
	6	Existen	ce of equilibria	234
	7	Regula	r economies	239
	8	Reduce	ed approaches	243
		8.1	Existence of equilibria and regular economies in the extended and the reduced approaches	245
		8.1.1 E	quilibria in terms of aggregate excess demand function 249	
		8.1.2	Equilibria in terms of projection	250
		8.2 On	the relationship between reduced and extended approx 251	ıch
	9	Append	dices	252
		9.1	Relationship between Assumption u4 and u4'	252
		9.2	Equivalent maximization problems	253
9.	PRO	DUCTI	ON ECONOMIES	257
	1	Set up	of the model	257
	2	Existen	ce of equilibria	268
		2.1	The test economy	268
		2.2	The homotopy	272
	3	Regular	rity	282
10	. TIM	E, UNC	ERTAINTY AND	
	INCO	OMPLE	TE MARKETS	289
	1	A mod	el with time, uncertainty and complete markets	289
	2	A mode	el with financial markets	293
		2.1	General set-up of the model	293
		2.2	Complete financial markets	298
		2.3	Incomplete financial markets	302
11	. NUN	/IERAIR	REASSETS	307
	1	Set up	of the model	308
	2	The der	mand function	311
	3	Existen	ce of equilibria	314
	4	Regula	r economies	322
12	. NON	AINAL A	ASSETS	327
	1	Introdu	ction	327

1

	2	Exister	nce of equilibria	328
	3	Real in	ideterminacy	330
		3.1	Definition of nominal and real indeterminacy.	330
		3.2	A step-by-step procedure to conjecture the degree of	
			nominal and real indeterminacy	332
		3.2.1	The model with nominal assets	333
		3.2.2	The model with nominal assets and exogenous asset prices	334
		3.2.3	The model with numeraire assets	335
		3.2.4	The model with real assets	335
		3.2.5	The model with a type of mixed assets	336
		3.2.6	The model with real assets with the extended approach and a different normalization	337
	4	Releva	ince and robustness of the indeterminacy result	339
	•	4 1	Why is indeterminacy important?	339
1		4.2	Why is indeterminacy robust?	340
		4.2.1	Sunspots	340
		4.2.2	Assets other than nominal	341
		4.2.3	Endogeneizing yields	341
		4.2.4	Outside money	342
		4.2.5	Restricted participation	343
	5	Real ir	ndeterminacy	
		in the 1	nominal asset model	343
		5.1	Step 1. Relationship between numeraire and nominal equilibria	346
		5.2 Ste	ep 2. Two sufficient conditions for different equilibrium	1
			allocations	347
		5.3	Step 3. The span condition holds	348
		5.4	Step 4. Generically, the rank condition holds	349
		5.5	The theorem	353
13	. REA	AL ASSE	ETS	355
	1	Set up	of the model	356
	2	Effecti	ve and pseudo-equilibria	359
	3	Exister	nce of pseudo-equilibria	362
	4	Generi	c existence of equilibria	384
	5	Regula	ar economies	385
14	. RES	TRICT	ED PARTICIPATION	395
-	1	Set up	of the model	396
	-	p		

2	Existe	nce of equilibria	399		
3	Regul	ar economies	407		
15. P	LANNER	INTERVENTION			
ON THE MARKET OUTCOME 4					
1	Introduction				
2	Gener	Generic Pareto non-optimality			
	2.1	Numeraire assets	419		
	2.2	Nominal assets	422		
	2.3	Real assets	427		
	2.4	Restricted participation	427		
3	3 The planner intervention: tools and goals				
4	4 Planner intervention and solutions to a system of equations				
	4.1	Submersion approach	434		
	4.2	Vector optimization approach	436		
5	Condi	Conditions for planner effectiveness			
	5.1	Openness	441		
	5.2	Density	444		
	5.2.1	Perturbation technique	444		
	5.2.2	A rank condition for density	449		
6	Verifi	cation of conditions for planner effectiveness	453		
	6.1 \$	Step 1	453		
	6.2	Step 2	455		
	6.2.1	Case $l.c_{Xh}$ ^0 for every $h$ .	455		
	6.2.2	Case 2. $Cx_h = 0$ for every $h$	457		
	6.2.3	Case 3. There exists an <i>h</i> such that $c_{Xh} = 0$	459		