Numerical Methods for Nonsmooth Dynamical Systems

Applications in Mechanics and Electronics

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With 81 Figures and 4 Tables
Part I Formulations of Nonsmooth Dynamical Systems

2 Nonsmooth Dynamical Systems: A Short Zoology

2.1 Differential Inclusions

2.1.1 Lipschitzian Differential Inclusions

2.1.2 Upper Semi-continuous DIs and Discontinuous Differential Equations

2.1.3 The One-Sided Lipschitz Condition

2.1.4 Recapitulation of the Main Properties of DIs

2.1.5 Some Hints About Uniqueness of Solutions

2.2 Moreau's Sweeping Process and Unilateral DIs

2.2.1 Moreau's Sweeping Process

2.2.2 Unilateral DIs and Maximal Monotone Operators

2.2.3 Equivalence Between UDIs and other Formalisms

2.3 Evolution Variational Inequalities

2.4 Differential Variational Inequalities

2.5 Projected Dynamical Systems

2.6 Dynamical Complementarity Systems

2.6.1 Generalities

2.6.2 Nonlinear Complementarity Systems

2.7 Second-Order Moreau's Sweeping Process

2.8 ODE with Discontinuities

2.8.1 Order of Discontinuity

2.8.2 Transversality Conditions

2.8.3 Piecewise Affine and Piecewise Continuous Systems

2.9 Switched Systems

2.10 Impulsive Differential Equations

2.10.1 Generalities and Well-Posedness

2.10.2 An Aside to Time-Discretization and Approximation

2.11 Summary

3 Mechanical Systems with Unilateral Constraints and Friction

3.1 Multibody Dynamics: The Lagrangian Formalism

3.1.1 Perfect Bilateral Constraints

3.1.2 Perfect Unilateral Constraints

3.1.3 Smooth Dynamics as an Inclusion

3.2 The Newton–Euler Formalism

3.2.1 Kinematics

3.2.2 Kinetics

3.2.3 Dynamics

3.3 Local Kinematics at the Contact Points

3.3.1 Local Variables at Contact Points

3.3.2 Back to Newton–Euler's Equations

3.3.3 Collision Detection and the Gap Function Calculation
3.4 The Smooth Dynamics of Continuum Media ........................................... 131
  3.4.1 The Smooth Equations of Motion ................................................. 131
  3.4.2 Summary of the Equations of Motion ........................................... 135
3.5 Nonsmooth Dynamics and Schatzman's Formulation ................................. 135
3.6 Nonsmooth Dynamics and Moreau's Sweeping Process ............................. 137
  3.6.1 Measure Differential Inclusions ................................................ 137
  3.6.2 Decomposition of the Nonsmooth Dynamics .................................. 137
  3.6.3 The Impact Equations and the Smooth Dynamics ............................. 138
  3.6.4 Moreau's Sweeping Process ..................................................... 139
  3.6.5 Finitely Represented \( \mathcal{C} \) and the Complementarity Formulation .... 141
3.7 Well-Posedness Results ..................................................................... 143
3.8 Lagrangian Systems with Perfect Unilateral Constraints: Summary ........... 143
3.9 Contact Models ................................................................................. 144
  3.9.1 Coulomb's Friction ................................................................. 145
  3.9.2 De Saxcé's Bipotential Function .............................................. 148
  3.9.3 Impact with Friction ............................................................... 151
  3.9.4 Enhanced Contact Models ......................................................... 153
3.10 Lagrangian Systems with Frictional Unilateral Constraints and Newton's Impact Laws: Summary .................................................. 161
3.11 A Mechanical Filippov's System ...................................................... 162

4 Complementarity Systems .................................................................. 165
4.1 Definitions ...................................................................................... 165
4.2 Existence and Uniqueness of Solutions .............................................. 167
  4.2.1 Passive LCS ............................................................................ 168
  4.2.2 Examples of LCS ..................................................................... 169
  4.2.3 Complementarity Systems and the Sweeping Process ................. 170
  4.2.4 Nonlinear Complementarity Systems ....................................... 172
4.3 Relative Degree and the Completeness of the Formulation ...................... 173
  4.3.1 The Single Input/Single Output (SISO) Case ............................. 174
  4.3.2 The Multiple Input/Multiple Output (MIMO) Case ..................... 175
  4.3.3 The Solutions and the Relative Degree ..................................... 175

5 Higher Order Constrained Dynamical Systems ..................................... 177
5.1 Motivations ...................................................................................... 177
5.2 A Canonical State Space Representation .......................................... 178
5.3 The Space of Solutions ..................................................................... 180
5.4 The Distribution \( \mathcal{D} \) and Its Properties ....................................... 180
  5.4.1 Introduction ............................................................................. 180
  5.4.2 The Inclusions for the Measures \( v_i \) ......................................... 182
  5.4.3 Two Formalisms for the HOSP .............................................. 183
  5.4.4 Some Qualitative Properties ................................................... 186
5.5 Well-Posedness of the HOSP .......................................................... 187
5.6 Summary of the Main Ideas of Chapters 4 and 5 ................................. 188
Part II Time Integration of Nonsmooth Dynamical Systems

7 Event-Driven Schemes for Inclusions with AC Solutions .......................... 203
    7.1 Filippov’s Inclusions ................................................................. 203
        7.1.1 Introduction ........................................................................... 203
        7.1.2 Stewart’s Method ................................................................ 205
        7.1.3 Why Is Stewart’s Method Superior to Trivial Event-Driven
              Schemes? .................................................................................. 213
    7.2 ODEs with Discontinuities with a Transversality Condition .......... 215
        7.2.1 Position of the Problem ......................................................... 215
        7.2.2 Event-Driven Schemes ............................................................ 215

8 Event-Driven Schemes for Lagrangian Systems ................................. 219
    8.1 Introduction .................................................................................. 219
    8.2 The Smooth Dynamics and the Impact Equations ......................... 221
    8.3 Reformulations of the Unilateral Constraints at Different
        Kinematics Levels .......................................................................... 222
        8.3.1 At the Position Level .............................................................. 222
        8.3.2 At the Velocity Level .............................................................. 222
        8.3.3 At the Acceleration Level ....................................................... 223
        8.3.4 The Smooth Dynamics ............................................................ 224
    8.4 The Case of a Single Contact ......................................................... 225
        8.4.1 Comments .............................................................................. 227
    8.5 The Multi-contact Case and the Index Sets .................................... 229
        8.5.1 Index Sets .............................................................................. 229
    8.6 Comments and Extensions ............................................................. 230
        8.6.1 Event-Driven Algorithms and Switching Diagrams .............. 230
        8.6.2 Coulomb’s Friction and Enhanced Set-Valued Force Laws .... 231
        8.6.3 Bilateral or Unilateral Dynamics? ......................................... 232
        8.6.4 Event-Driven Schemes: L"{o}fstedt’s Algorithm .................. 232
        8.6.5 Consistency and Order of Event-Driven Algorithms .......... 236
    8.7 Linear Complementarity Systems .................................................... 240
    8.8 Some Results ................................................................................ 241
9 Time-Stepping Schemes for Systems with AC Solutions

9.1 ODEs with Discontinuities

9.1.1 Numerical Illustrations of Expected Troubles

9.1.2 Consistent Time-Stepping Methods

9.2 DIs with Absolutely Continuous Solutions

9.2.1 Explicit Euler Algorithm

9.2.2 Implicit $\theta$-Method

9.2.3 Multistep and Runge–Kutta Algorithms

9.2.4 Computational Results and Comments

9.3 The Special Case of the Filippov’s Inclusions

9.3.1 Smoothing Methods

9.3.2 Switched Model and Explicit Schemes

9.3.3 Implicit Schemes and Complementarity Formulation

9.3.4 Comments

9.4 Moreau’s Catching-Up Algorithm for the First-Order

9.4.1 Mathematical Properties

9.4.2 Practical Implementation of the Catching-up Algorithm

9.4.3 Time-Independent Convex Set $K$

9.5 Linear Complementarity Systems with $r \leq 1$

9.6 Differential Variational Inequalities

9.6.1 The Initial Value Problem (IVP)

9.6.2 The Boundary Value Problem

9.7 Summary of the Main Ideas

10 Time-Stepping Schemes for Mechanical Systems

10.1 The Nonsmooth Contact Dynamics (NSCD) Method

10.1.1 The Linear Time-Invariant Nonsmooth Lagrangian Dynamics

10.1.2 The Nonlinear Nonsmooth Lagrangian Dynamics

10.1.3 Discretization of Moreau’s Inclusion

10.1.4 Sweeping Process with Friction

10.1.5 The One-Step Time-Discretized Nonsmooth Problem

10.1.6 Convergence Properties

10.1.7 Bilateral and Unilateral Constraints

10.2 Some Numerical Illustrations of the NSCD Method

10.2.1 Granular Material

10.2.2 Deep Drawing

10.2.3 Tensegrity Structures

10.2.4 Masonry Structures

10.2.5 Real-Time and Virtual Reality Simulations

10.2.6 More Applications

10.2.7 Moreau’s Time-Stepping Method and Painlevé Paradoxes

10.3 Variants and Other Time-Stepping Schemes

10.3.1 The Paoli–Schatzman Scheme

10.3.2 The Stewart–Trinkle–Anitescu–Potra Scheme
11 Time-Stepping Scheme for the HOSP ........................................ 319
  11.1 Insufficiency of the Backward Euler Method ................. 319
  11.2 Time-Discretization of the HOSP ........................................... 321
    11.2.1 Principle of the Discretization ................................ 321
    11.2.2 Properties of the Discrete-Time Extended Sweeping Process 322
    11.2.3 Numerical Examples ............................................ 324
  11.3 Synoptic Outline of the Algorithms ................................. 325

Part III Numerical Methods for the One-Step Nonsmooth Problems

12 Basics on Mathematical Programming Theory ....................... 331
  12.1 Introduction ............................................. 331
  12.2 The Quadratic Program (QP) ........................................... 331
    12.2.1 Definition and Basic Properties ................................ 331
    12.2.2 Equality-Constrained QP ..................................... 335
    12.2.3 Inequality-Constrained QP .................................. 338
    12.2.4 Comments on Numerical Methods for QP ....................... 344
  12.3 Constrained Nonlinear Programming (NLP) ......................... 345
    12.3.1 Definition and Basic Properties ............................... 345
    12.3.2 Main Methods to Solve NLPs .................................... 347
  12.4 The Linear Complementarity Problem (LCP) ...................... 351
    12.4.1 Definition of the Standard Form ................................ 351
    12.4.2 Some Mathematical Properties ................................. 352
    12.4.3 Variants of the LCP ........................................... 355
    12.4.4 Relation Between the Variants of the LCPs .................. 357
    12.4.5 Links Between the LCP and the QP .............................. 359
    12.4.6 Splitting-Based Methods ..................................... 363
    12.4.7 Pivoting-Based Methods ...................................... 367
    12.4.8 Interior Point Methods ....................................... 374
    12.4.9 How to chose a LCP solver? .................................... 379
  12.5 The Nonlinear Complementarity Problem (NCP) ................... 379
    12.5.1 Definition and Basic Properties ............................... 379
    12.5.2 The Mixed Complementarity Problem (MCP) ................. 383
    12.5.3 Newton–Josephy’s and Linearization Methods ............... 384
    12.5.4 Generalized or Semismooth Newton’s Methods ................ 385
    12.5.5 Interior Point Methods ....................................... 388
    12.5.6 Effective Implementations and Comparison of the Numerical Methods for NCPs ................................. 388
  12.6 Variational and Quasi-Variational Inequalities .................. 389
    12.6.1 Definition and Basic Properties ............................... 389
    12.6.2 Links with the Complementarity Problems ................ 390
    12.6.3 Links with the Constrained Minimization Problem ........ 391
    12.6.4 Merit and Gap Functions for VI ............................... 392
12.6.5 Nonsmooth and Generalized equations ........................................ 396
12.6.6 Main Types of Algorithms for the VI and QVI ............................ 398
12.6.7 Projection-Type and Splitting Methods ...................................... 398
12.6.8 Minimization of Merit Functions ............................................. 400
12.6.9 Generalized Newton Methods ................................................ 401
12.6.10 Interest from a Computational Point of View ............................. 401
12.7 Summary of the Main Ideas ....................................................... 401

13 Numerical Methods for the Frictional Contact Problem ....................... 403
13.1 Introduction ............................................................................. 403
13.2 Summary of the Time-Discretized Equations .................................. 403
13.2.1 The Index Set of Forecast Active Constraints ............................ 403
13.2.2 Summary of the OSNSPs ....................................................... 405
13.3 Formulations and Resolutions in LCP Forms .................................. 407
13.3.1 The Frictionless Case with Newton's Impact Law ......................... 407
13.3.2 The Frictionless Case with Newton's Impact and Linear Perfect Bilateral Constraints .................................................. 408
13.3.3 Two-Dimensional Frictional Case as an LCP ............................... 409
13.3.4 Outer Faceting of the Coulomb's Cone .................................... 410
13.3.5 Inner Faceting of the Coulomb's Cone .................................... 414
13.3.6 Comments ................................................................. 417
13.3.7 Weakness of the Faceting Process ......................................... 418
13.4 Formulation and Resolution in a Standard NCP Form ....................... 419
13.4.1 The Frictionless Case ......................................................... 419
13.4.2 A Direct MCP for the 3D Frictional Contact ............................. 419
13.4.3 A Clever Formulation of the 3D Frictional Contact as an NCP ......... 420
13.5 Formulation and Resolution in QP and NLP Forms ........................ 422
13.5.1 The Frictionless Case ......................................................... 422
13.5.2 Minimization Principles and Coulomb's Friction ........................ 423
13.6 Formulations and Resolution as Nonsmooth Equations ........................ 424
13.6.1 Alart and Curnier's Formulation and Generalized Newton's Method ................................................................. 424
13.6.2 Variants and Line-Search Procedure ..................................... 429
13.6.3 Other Direct Equation-Based Reformulations ............................. 430
13.7 Formulation and Resolution as VI/CP ......................................... 432
13.7.1 VI/CP Reformulation .......................................................... 432
13.7.2 Projection-type Methods ...................................................... 433
13.7.3 Fixed-Point Iterations on the Friction Threshold and Ad Hoc Projection Methods .................................................. 434
13.7.4 A Clever Block Splitting: the Nonsmooth Gauss–Seidel (NSGS) Approach ................................................................. 437
13.7.5 Newton's Method for VI ...................................................... 440
Part IV The SICONOS Software: Implementation and Examples

14 The SICONOS Platform ........................................ 443
  14.1 Introduction ............................................ 443
  14.2 An Insight into SICONOS ................................. 443
    14.2.1 Step 1. Building a Nonsmooth Dynamical System .... 444
    14.2.2 Step 2. Simulation Strategy Definition ............ 447
  14.3 SICONOS Software ....................................... 448
    14.3.1 General Principles of Modeling and Simulation ...... 448
    14.3.2 NSDS-Related Components ........................... 451
    14.3.3 Simulation-Related Components ....................... 456
    14.3.4 SICONOS Software Design ........................... 457
  14.4 Examples .............................................. 460
    14.4.1 The Bouncing Ball(s) ............................... 460
    14.4.2 The Woodpecker Toy ................................ 463
    14.4.3 MOS Transistors and Inverters ...................... 464
    14.4.4 Control of Lagrangian systems ..................... 466

A Convex, Nonsmooth, and Set-Valued Analysis .................... 475
  A.1 Set-Valued Analysis ..................................... 475
  A.2 Subdifferentiation ....................................... 475
  A.3 Some Useful Equivalences ................................ 476

B Some Results of Complementarity Theory .......................... 479

C Some Facts in Real Analysis .................................. 481
  C.1 Functions of Bounded Variations in Time ................. 481
  C.2 Multifunctions of Bounded Variation in Time ............ 482
  C.3 Distributions Generated by RCLSBV Functions ............ 483
  C.4 Differential Measures .................................. 486
  C.5 Boehr's Distributions .................................... 487
  C.6 Some Useful Results .................................... 487

References .................................................... 489

Index .......................................................... 519