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Critical Phenomena in Natural Sciences

Chaos, Fractals,
Selforganization and Disorder:
Concepts and Tools

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With 102 Figures

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